# 1D Coaxial Photonic Crystal - Superluminal

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J. N. Munday and W. M. Robertson have reported observing: *Negative group velocity pulse tunneling through a coaxial photonic crystal*, Applied Physics Letters, Sept. 2002. In the following, we analyze this situation using Mathcad.

When a signal traverses an impedance boundary, it experiences a phase shift and a partial reflection that can be calculated from the generalized optical Fresnel coefficients of reflection r (|Vreflected/Vincident|) and transmission t |Vtransmitted/Vincident|). This can be generalized to coaxial cables using characteristic impedance, where zi, zr and zt are the impedance of the incident, refelected and transmitted media.

$$r = \frac{zi - zt}{zi + zt} \qquad t = \frac{2 \cdot zi}{zi + zt}$$

Periodic variation in the impedance of a medium can produce destructive interferrence for some wavelengths.The phase accumulated throughout the crystal changes rapidly with frequency, especially near the band gap.

A unit cell consists of two coax segments, one of 50 ohm RG-58/U and one with 75 ohm RG-59/U. Each segment has the same phase velocity 0.66c and length 8 ft. As a result of impedance mismatch, 20% of the field is reflected at each interface. 12 unit cells of total length 120 m are used. A deep stop gap between 18 and 23 MHz occurs. Outside gap the attenuation is 25 - 35 dB/km. Stop occurs when path length of unit cell is multiple of l/2.

To calculate dispersive properties and group velocity, the effective index theory is used. The theory says phase shift and scattering loss of the electric field is from an effective complex index of refraction. the real part of the index, nr, is obtained from the overall phase shift f accumulated throught the crystal of length D. t is the complex coefficient of electric field transmission over the whole crystal and m = 0

$$n := 0 \qquad \phi = \arctan\left(\frac{Im(t)}{Re(t)}\right) + n \cdot \pi \qquad n_r(\omega) = \frac{c \cdot \phi}{\omega \cdot D} \qquad f_{stop} = \frac{v}{2d}$$

We transmit a sinusoidal carrier with a Gaussian shaped pulse envelope. For carrier of 5 - 15 MHz pulse duration was scaled from 6 to 2 us, keeping the number of cycles within the envelope constant at 30 while varying the bandwidth fro 0.15 to 0.45 MHz.

## COAXIAL CABLE MODEL

Number of Sections, N

$l58 := 20 \cdot m \qquad l59 := 15 \cdot m$

### Physical Constants

$$c := 299792458 \cdot \frac{m}{sec} \qquad \mu_o := 4 \cdot \pi \cdot 10^{-7} \cdot \frac{newton}{amp^2} \qquad \varepsilon_o := 8.854187817 \cdot 10^{-12} \cdot \frac{farad}{m}$$

$$\rho_{cu} := 1.673 \cdot 10^{-6} \cdot \Omega \cdot cm \qquad \sigma := \rho_{cu}^{-1} \qquad \mu_r := 1 \qquad \mu := \mu_o \cdot \mu_r \qquad \lambda o(f) := \frac{c}{f}$$

### RG-58/U (50 Ω) and RG-59/U (75 Ω) Coaxial Cable Data:

εr is the dielectric constant Ydb is attenuation in dB

$$Crg58U_{len} := 95 \cdot \frac{pf}{m} \qquad a := \frac{1}{2} \cdot 0.9mm \qquad b := \frac{1}{2} \cdot 2.9 \cdot mm \qquad \varepsilon_r := 2.29 \qquad v\_c := 0.66 \quad l58 := 8 \cdot ft$$

dB_100m 5,55,500MHz 0.86,2, 5.7

$$Crg59U_{len} := 68.6 \cdot \frac{pf}{m} \qquad a_{59} := \frac{1}{2} \cdot 0.81mm \quad b_{59} := \frac{1}{2} \cdot 3.66 \cdot mm \qquad v\_c59 := 0.66 \quad l59 := 8 \cdot ft$$

$$Crg62U_{len} := \blacksquare \cdot \frac{pf}{m} \qquad a_{59} := \frac{1}{2} \cdot 0.81mm \quad b_{59} := \frac{1}{2} \cdot 3.66 \cdot mm \qquad v\_c62 := 0.85 \quad l62 := 7.97 \cdot m$$

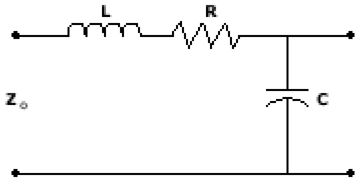
$$\varepsilon 58 := \varepsilon_o \cdot \varepsilon_r \qquad d := l58 + l59$$

$$Ydb_{Rg58}(f_{MHz}) := 1.292 + 1.537 \cdot \sqrt{f_{MHz}} + 0.0157 \cdot f_{MHz} \qquad \tan\delta := 0.0004 \qquad f_{gap} := \frac{v\_c \cdot c}{2 \cdot d}$$

$$\varepsilon 59_r := \frac{1}{v\_c59^2} \qquad \varepsilon 59 := \varepsilon_o \cdot \varepsilon 59_r$$

**Coaxial Cable Model**

Conductor Radii a and b




$$C58_{Len} := \frac{2\cdot\pi\cdot\varepsilon58}{\ln\left(\frac{b}{a}\right)} \qquad C59_{Len} := \frac{2\cdot\pi\cdot\varepsilon59}{\ln\left(\frac{b_{59}}{a_{59}}\right)}$$

$$\delta_{skin}(\omega) := \frac{1}{\sqrt{\pi\cdot\frac{\omega}{2\cdot\pi}\cdot\mu_o\cdot\mu_r\cdot\sigma}}$$

$$L58_{Len} := \frac{\mu}{2\cdot\pi}\cdot\ln\left(\frac{b}{a}\right) \qquad L59_{Len} := \frac{\mu}{2\cdot\pi}\cdot\ln\left(\frac{b_{59}}{a_{59}}\right) \qquad \sqrt{\frac{1}{\varepsilon_r}} = 0.661$$

$$R58_{Len}(\omega) := \frac{a + b}{2\cdot\pi\cdot a\cdot b\cdot\delta_{skin}(\omega)\cdot\sigma} \qquad R59_{Len}(\omega) := \frac{a_{59} + b_{59}}{2\cdot\pi\cdot a_{59}\cdot b_{59}\cdot\delta_{skin}(\omega)\cdot\sigma} \qquad \varepsilon_{r\lambda} = \left(\frac{\lambda_{coax}}{\lambda_o}\right)^2$$

p= r  Return Loss $= 10\ \mathrm{Log}_{10}\left(\frac{1}{1-p^2}\right)$ dB  $\lambda(f) := c\cdot\left(\sqrt{\varepsilon}\cdot f\right)$  $v = \frac{1}{\sqrt{L_{len}\cdot C_{len}}} = \frac{2\cdot Length}{\Delta t}$

$$f_{gap} = 2.029\times10^{7}\ \frac{1}{s}$$

**Characteristic Impedance**

**(a is conductor and b is dielectric radii, Llen and Clen are L and C per length. ):**

$$Z_o(\omega) := \sqrt{\frac{R + j\cdot\omega\cdot L}{G + j\cdot\omega\cdot C}} \qquad Z58_{olc} := \sqrt{\frac{L58_{Len}}{C58_{Len}}} \qquad Z_o := 138\cdot\sqrt{\frac{1}{\varepsilon59_r}}\cdot\log\left(\frac{b_{59}}{a_{59}}\right) \qquad Z_o = 59.657$$

Attenuation, Phase and Propogation Constants: $\alpha, \beta, \gamma$  $Z58_{olc} = 46.36\,\Omega$

$$\gamma(\omega) = \alpha(\omega) + j\cdot\beta(\omega) \qquad \gamma(\omega) := \sqrt{(R + j\cdot\omega\cdot L)(G + j\cdot\omega\cdot C)} \qquad \alpha(\omega) := \mathrm{Re}(\gamma(\omega)) \qquad \beta(\omega) := \mathrm{Im}(\gamma(\omega))$$

$$E(\omega, x) = E_o\cdot e^{-(\alpha(\omega)+j\cdot\beta(\omega))\,x} \qquad V(z) = V_{incident}\cdot e^{-\gamma\cdot z} + V_{reflected}\cdot e^{\gamma\cdot z} \qquad \alpha_s(f) := 2\cdot\beta(2\cdot\pi\cdot f)$$

$$AbsorbCoef(f) := 10^{-7}\cdot\left(2\cdot\pi\cdot\frac{\sqrt{f}}{m}\right) \qquad \kappa(f) := \sqrt{\mu_o\cdot\varepsilon_o\cdot\varepsilon_r}\cdot2\cdot\pi\cdot\frac{f}{sec} + j\cdot\frac{AbsorbCoef(f)}{2}$$ wave number

$$E(x) = E_o\cdot e^{j\cdot(\kappa\cdot x+\phi)}$$

Material Loss tangent, tanδ equals  jG_ωC  Thus  $G = j\cdot\omega\cdot C\cdot\tan\delta = 2\,\frac{\pi\cdot\varepsilon_o\cdot\varepsilon_r\cdot length\cdot\omega\cdot\tan\delta}{\ln\left(\frac{b}{a}\right)}$

$$G58(\omega) := j\cdot\omega\cdot Crg58U_{len}\cdot\tan\delta \qquad G59(\omega) := j\cdot\omega\cdot Crg59U_{len}\cdot\tan\delta$$

$$Z58_o(f) := \sqrt{\frac{R58_{Len}\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot L58_{Len}}{G58\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot Crg58U_{len}}} \qquad Z59_o(f) := \sqrt{\frac{R59_{Len}\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot L59_{Len}}{G59\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot Crg59U_{len}}}$$

**Reflection and Transmission Coefficients, t and r**

From Maxwells Equations: Plane incident E-M wave traveling from medium A to B

$$t_{AB}(f) := \frac{2\cdot Z59_o(f)}{Z58_o(f) + Z59_o(f)} \qquad t_{BA}(f) := \frac{2\cdot Z58_o(f)}{Z58_o(f) + Z59_o(f)}$$

$$r_{AB}(f) := \frac{Z59_o(f) - Z58_o(f)}{Z58_o(f) + Z59_o(f)} \qquad r_{BA}(f) := \frac{Z58_o(f) - Z59_o(f)}{Z58_o(f) + Z59_o(f)}$$

Structure of Lengths of N unit cells starting with Len58 $N := 12$

$j := 0..N \qquad Len58_j := l58 \qquad Len58_0 := l58 \qquad Len59_j := l59 \qquad Len59_0 := 0 \cdot m \qquad D := N(d)$

$$M_{BA}(f,j) := \begin{pmatrix} \dfrac{e^{-j\cdot\kappa(f)\cdot Len58_j}}{t_{BA}(f)} & \dfrac{r_{BA}(f)\cdot e^{j\cdot\kappa(f)\cdot Len58_j}}{t_{BA}(f)} \\ \dfrac{r_{BA}(f)\cdot e^{-j\cdot\kappa(f)\cdot Len58_j}}{t_{BA}(f)} & \dfrac{e^{j\cdot\kappa(f)\cdot Len58_j}}{t_{BA}(f)} \end{pmatrix}$$

$$M_{AB}(f,j) := \begin{pmatrix} \dfrac{e^{-j\cdot\kappa(f)\cdot Len59_j}}{t_{AB}(f)} & \dfrac{r_{AB}(f)\cdot e^{j\cdot\kappa(f)\cdot Len59_j}}{t_{AB}(f)} \\ \dfrac{r_{AB}(f)\cdot e^{-j\cdot\kappa(f)\cdot Len59_j}}{t_{AB}(f)} & \dfrac{e^{j\cdot\kappa(f)\cdot Len59_j}}{t_{AB}(f)} \end{pmatrix}$$

## Field Amplitudes, E, for Transmitted and Reflected Waves

$$E(f) := \left[\prod_{j=1}^{N} \left(M_{AB}\left(f\cdot 10^6, j\right)\cdot M_{BA}\left(f\cdot 10^6, j-1\right)\right)\right]\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

**Phase $\phi$ - Effective Index Theory**

$$\phi\left(f_{MHz}\right) := -\arg\left[\left(E\left(f_{MHz}\right)_0\right)^{-1}\right]$$

$$T_s\left(f_{MHz}\right) := \left|\left(E\left(f_{MHz}\right)_0\right)^{-1}\right|$$

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Phz := | G ← 0
       | P_1 ← φ(1/10)
       | Pold ← P_1
       | for n ∈ 2..150
       |     | Pn ← φ(n/10)
       |     | G ← G + 1  if  Pn > Pold + 2
       |     | Pold ← Pn
       |     | P_n ← Pn − G·π
       | P
```

**Group time lag, tg, Group Velocity, vg**

$$\tau_g(f) := -\left(\frac{1}{2\cdot\pi\cdot\dfrac{10^6}{sec}}\cdot\frac{\phi(f+0.01)-\phi(f-0.01)}{0.02}\right)$$

$$v_g(f) := \frac{D}{c\cdot\tau_g(f)}$$

WRITEPRN("PhaseBrk") := Phz

$\phi_{break}$ := READPRN("PhaseBrk.prn")

## Calculate Complex Vector Arrays to Store Results for Plotting

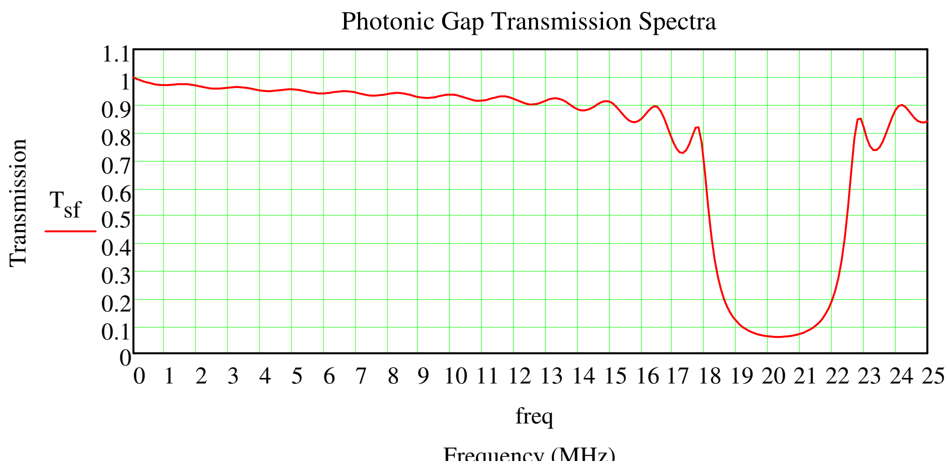
$n := 1..250 \qquad freq_n := \dfrac{n}{10} \qquad E_{f_n} := E\left(freq_n\right) \qquad T_{sf_n} := \left|\left[\left(E_{f_n}\right)_0\right]^{-1}\right| \qquad \phi_{f_n} := -\arg\left[\left[\left(E_{f_n}\right)_0\right]^{-1}\right]$

$$\tau_{gf_n} := -\left(\frac{1}{2\cdot\pi\cdot\dfrac{10^6}{sec}}\cdot\frac{\phi_{f_n}-\phi_{f_{n-1}}}{0.1}\right) \qquad v_{gf_n} := \frac{D}{c\cdot\tau_{gf_n}}$$

$c_{vacuum} := 1$

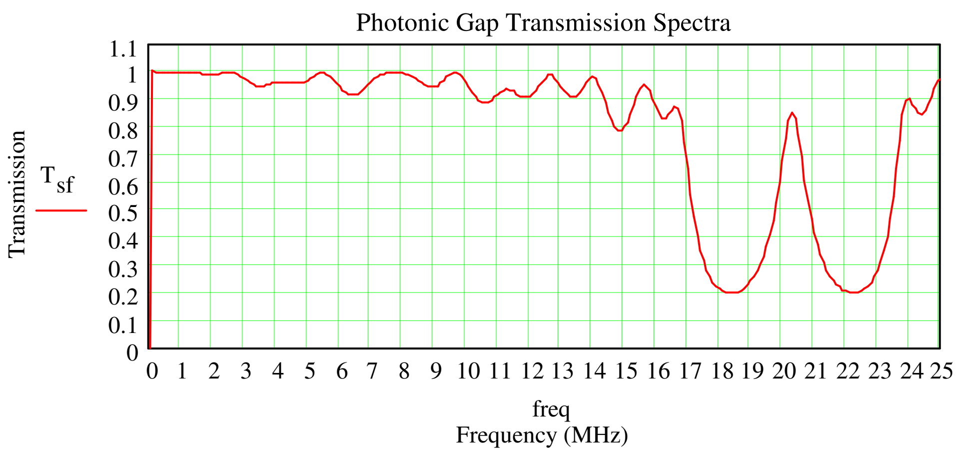
$f_{gap} = 2.029\times 10^{7}\,\dfrac{1}{s}$

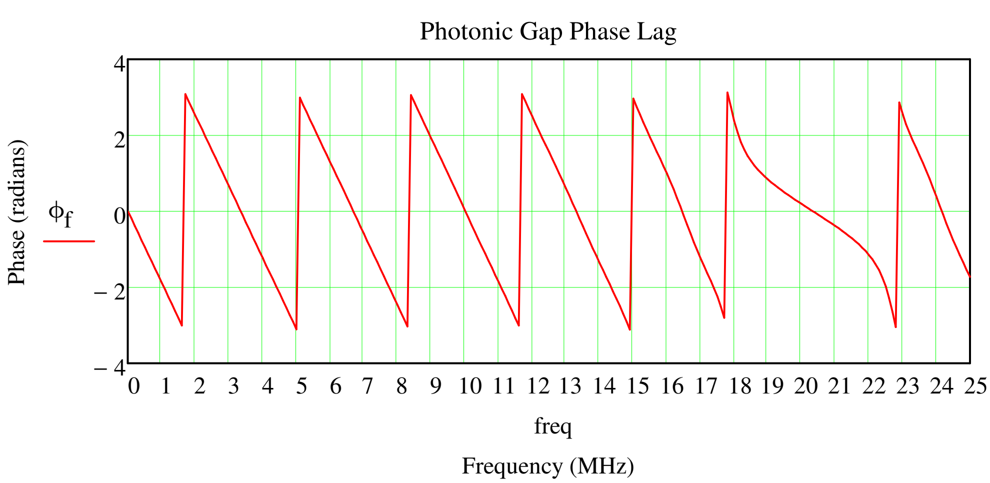
$T_{sf_0} := 1 \qquad dbT := \overrightarrow{\left(10\cdot\log\left(T_{sf}\right)\right)}$

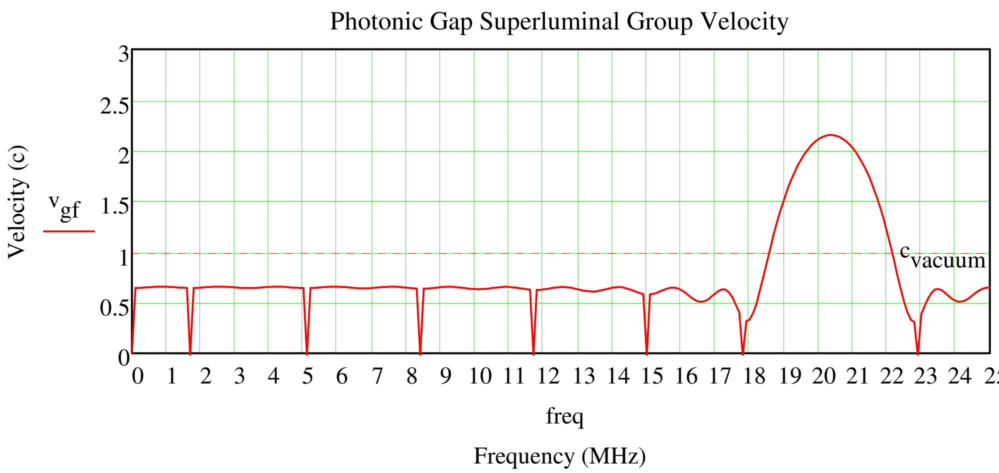



$$\text{match}\left(\min\left(T_{sf}\right), T_{sf}\right)^{T} \cdot 0.1 = \left(20.1 \quad 20.2 \quad 20.3 \quad 20.4 \quad 20.5\right)$$

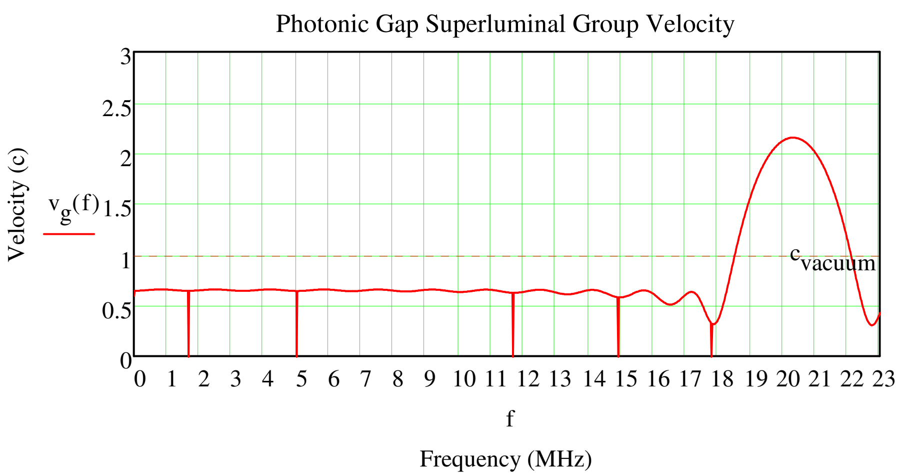
**Result (picture) for** $Len_{58_2} := 4 \cdot 158$ $Len_{58_5} := 4 \cdot 158$



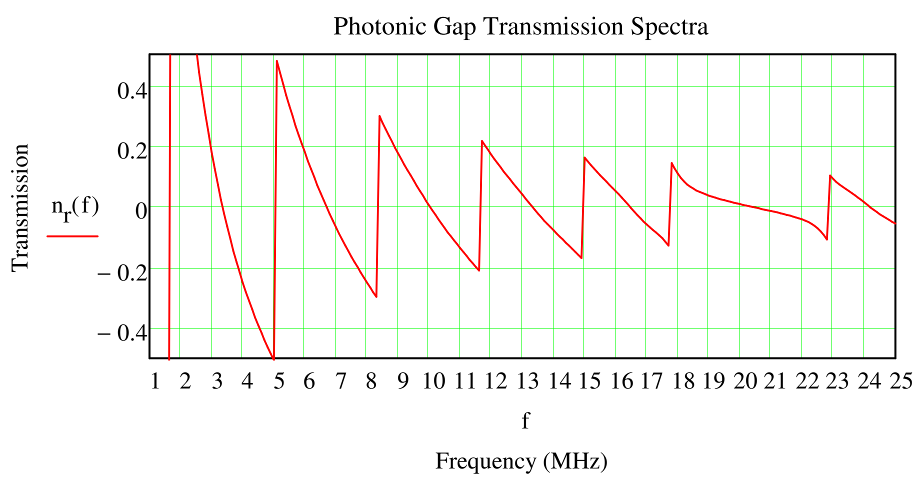



**Below are exact solutions. Evaluation was disabled because they take a long time to calculate each function and then repaint each of the graphs**




$$\Delta t := \frac{12\cdot(158 + 159)}{0.66\cdot c} \qquad \Delta t = 295.768\ \text{nsec}$$

$$n_r(f) := \frac{c\cdot\phi(f)}{2\pi\cdot\frac{f}{\text{sec}}\cdot 10^6\cdot D} \qquad n_r(1) = -1.526 \qquad f := 1, 1.1 .. 25$$



## Impedance Model

$$\gamma58(f) := \sqrt{\left(R58_{Len}\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot L58_{Len}\right)\left(G58\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot Crg58U_{len}\right)}$$

$$\gamma59(f) := \sqrt{\left(R59_{Len}\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot L59_{Len}\right)\left(G59\left(\frac{2\cdot\pi\cdot f}{sec}\right) + j\cdot\frac{2\cdot\pi\cdot f}{sec}\cdot Crg59U_{len}\right)}$$

$$Z58(Z_L, f) := Z58_o(f)\cdot\left(\frac{Z_L\cdot\cosh(\gamma58(f)\cdot l58) + Z58_o(f)\cdot\sinh(\gamma58(f)\cdot l58)}{Z58_o(f)\cdot\cosh(\gamma58(f)\cdot l58) + Z_L\cdot\sinh(\gamma58(f)\cdot l58)}\right)$$

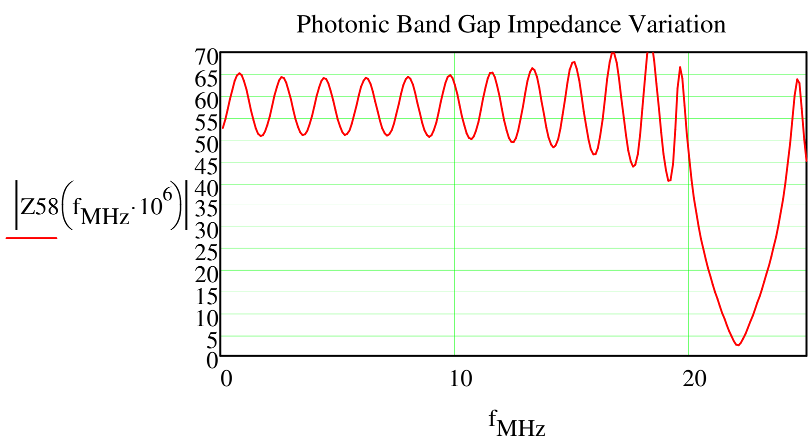
$$Z59(Z_L, f) := Z59_o(f)\cdot\left(\frac{Z_L\cdot\cosh(\gamma59(f)\cdot l59) + Z59_o(f)\cdot\sinh(\gamma59(f)\cdot l59)}{Z59_o(f)\cdot\cosh(\gamma59(f)\cdot l59) + Z_L\cdot\sinh(\gamma59(f)\cdot l59)}\right)$$

## 12 Alternating RG58/59 Sections Cable

$$Z58i(f) := Z58(Z59(Z58(Z59(Z58(Z59(Z58(Z59(Z58(Z59(Z58(Z59(50\cdot\Omega, f), f), f), f), f), f), f), f), f), f), f), f)$$

$$Z58(f) := Z58(Z59(Z58(Z59(Z58(Z59(Z58(Z59(Z58(Z59(Z58(Z59(Z58i(f), f), f), f), f), f), f), f), f), f), f), f)$$

$$f_{MHz} := 0, 0.1 .. 25$$




VSWR:

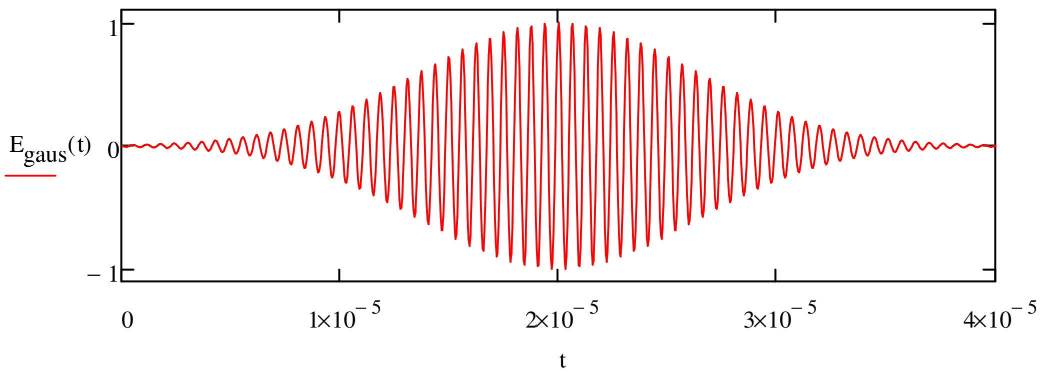
The traditional way to determine the reflection coefficient is to measure the standing wave caused by the superposition of the incident wave and the reflected wave. Traditionally the voltage is measured at a series of points using a slotted line. The ratio of the maximum divided by the minimum is the Voltage Standing Wave Ratio (VSWR). The VSWR is infinite for total reflections because the minimum voltage is zero. If no reflection occurs the VSWR is 1.0. VSWR and reflection coefficient are related as follows:

$$VSWR = (1 + \rho)/(1 - \rho)$$

Multiple ReflectionsIf there is a series of impedance changes, each one will cause a reflection. The total reflection is the vector addition of each of the individual coefficients accounting for the distance between discontinuities. Even though the calculations are difficult, a total VSWR can still be measured.

**Type of Pulse Wave Packet for Transmission into Photonic Crystal Cable**

$$\omega_o := 1\cdot 10^{7} \qquad \tau := 10\cdot\frac{2\cdot\pi}{\omega_o} \qquad \phi_g := 2\cdot 10^{-5} \qquad E_{gaus}(t) := \mathrm{Re}\left[e^{\frac{-(t-\phi_g)^2}{2\cdot\tau^2}}\cdot e^{-j\cdot\omega_o\cdot(t-\phi_g)}\right]$$

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