Analysis & Simulation of Apollo 11 Moon Flight

This review and analysis were created
both as a tribute and a commemoration
of the 50th anniversary of the historic Apollo 11 Flight to the Moon.
I had the good fortune to work on the development of the
inertial guidance system for this program. Our goal is to recreate the
journey of the Apollo vessel from earth to moon and back again.
We will do this by solving for the trajectories and
reviewing and simulating the rocket burns of the Apollo spacecraft.
Our analysis starts on page 18.

As an introduction to this analysis,
we also provide the historic background and context,
review the strategies, rocket fuel burn specs,
and orbit parameters required for the trip.

The Historic Apollo Flight to the Moon

The Apollo Space Program is regarded by many as one of the greatest engineering and scientific accomplishments of all time. At its peak, it employed 400,000 people and its budget represented 2.2% of all federal spending. On **July 16, 1969**, the Apollo 11 Saturn IV Spacecraft blasted off from Cape Canaveral, FL, and four days later, the first man walked on the moon. Since that time, roughly 50 years ago, only twelve men have walked on the surface of the moon. After obtaining my MS in Physics from the University of Wisconsin, I had the good fortune to work on the Apollo Space Program as a Reliability Project Engineer starting on July 16, 1966.

I worked on the development of the Inertial Guidance System and Computer at the AC Electronics Facility located in Oak Creek, Wisconsin. The Apollo Guidance System (AGS), consisted of the digita Apollo Guidance Computer (AGC) and the Inertial Measurement Unit (IMU), an assembly of a three-gimbaled stable platform with three corresponding accelerometers positioned in roll, pitch, and yaw. The were two AGSs used for guidance, one each in the Command and the Lunar Excursion Module (LEM). In my 37 year career as a Senior Physicist, this was the most exciting project that I have ever worked on. It was a project that had the attention of the whole world. Personally, one of the things that I get a kick about concerning my work on Apollo, is the fact that:
the LEM that I helped develop still has some of its parts left on the surface of the moon.

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January 14, 2019
Tom Kotowski

BACKGROUND

Why did we choose to go to the Moon?

In 1957, as part of the International Geophysical Year, the USSR launched the first ever artificial satellite, Sputnik 1. It was launched with an R-7 missile, the world's first intercontinental ballistic missile (ICBM). Sputnik orbited the earth emitting signals at frequencies that anyone with an amateur radio could listen. The launch of Sputnik marked the unofficial start of the space race between the US and the USSR.

Initially, the US responded by lauching a rocket of its own, the Vanguard TV3. However, it blew up on the launch pad. Coupled with the very public failure of Vanguard TV3, the US desperately needed to get out of the starting block in the space race. The US finally got off the ground with the launch of Explorer 1 on February of 1958.

The first ICBMs were deployed by the Soviet Union in 1958; the United States followed the next year, and China some 20 years later. The first principal U.S. ICBM was the silo-launched Minuteman missile. On 12 April 1961, the USSR launched the first man into low earth orbit for one trip around the earth. In response to this Soviet space challenge, on May 25, 1961, President John F. Kennedy announced his goal of putting a man on the moon by the end of the decade.

After 50 years, why have we not gone back to the Moon?

Shortly after Apollo 11, there were marked world changes, such as the Vietnam war and detente with the Russians. As a result from these changes, and a lessening of public, political, and military interest and will for space travel, there was a sharp decline in Apollo and Aerospace funding. On September 2, 1970, as a result of federal budget cuts, NASA announced it was canceling the Apollo H4 and J4 missions. In the next few years, Apollo funding dropped by 50%. The flight of Apollo 17 in late 1972 would bring the program to a close. This was the last time that man has set foot on the moon.

Despite huge gains in some areas of technology, such as computers, the Apollo liquid fuel rocket technology still remains the state of the art. NASA is presently working on developing the Space Launch System (SLS) which is capable of going to the moon and beyond to Mars. At present, the SLS is over budget and delayed. It has an initial rocket thrust of 53 million N, sufficient to get a Saturn V size payload to the moon. The SLS could be used to lift the four man Orion space capsule to the moon. SLS testing will not be ready until 2022.

Goal of this Analysis:

Recreate the History & Dynamics of the Apollo Flight to the Moon & Back

Motivated by the upcoming ***50th anniversary of the first Apollo moon flight***, my goal in this exercise is to do short review of the basics of the dynamics of space flight and to simulate the basic flight and orbital dynamics and trajectories of the first manned mission to the moon and back. A crucial element in this goal is to **build "toy" orbital models** that interested parties and I can play with. These models are implemented in Mathcad 14. A trajectory is a path or curve of an object as it evolves along its path in time and space. It can be specified by a mathematical model, formula, or a set of points.

This is an Outline of the Approach We Will Use in the Analytic Part of this Work:

Two Different Perspective:
We will adopt two different perspectives and types of investigations in our study. The first is the broad viewpoint of the **Mission Designer**, where we start with a clean sheet and just look at various solutions to get to the moon and back. We will answer the question of what kinds of trajectories can get us to the moon and back. The second perspective is **Historical.** This is a more detailed and segmented. Given the specific rocket burns and conditions that were taken by Apollo 11 to the moon and back, calculate the resultant accelerations, burn velocities, and positions for each rocket engine stage and compare them to the historical values.

Mission Designer Perspective:

To accomplish this, we want to develop a methodology that is powerful and general enough to solve a range of trajectory and AstroDynamics problems. For example: Have the capability of figuring different types of trajectories to different destinations, such as the Moon or planets, such as Mars.

Determine the Apollo Mission to Moon and Back Trajectories (See Sections I, XVII, & XVIII)
After rocket burn, a precise trajectory can only be done on an-hour-by-hour basis for a given set of initial position and velocity conditions at the injection point, using numerical integration of the equations of motion. These calculations must be informed by data from a Lunar Ephemeris Table, using a 3-Body and 4-Body gravitational planar point mass system. Because of the complexity of the calculations, a trial and error method must be used to hone in on the desired trajectory. Approximate Analytic Algorithms can be used to narrow down the range of possible solution space sets. Once a candidate trajectory is obtained, optimization techniques can be used to minimize fuel costs or optimize other parameters of interest, such as length of time of flight.

Our task is to figure out trajectories, that is the actual path through time and space, that get us from the earth launch point to lunar destination point and then back to the earth splashdown point. We will affect this solution by solving the equations of motion for earth, moon, sun, and spaceship (4-body) and then using computer numerical computation methods. We will also investigation a number of different kinds of trajectories for the Apollo trip to the moon and back. For this task we use the Mathcad differential equation solver implementing either an Adams/BDF or Radu numerical integration method.

Computation
The design and in-flight main engine computations for Apollo orbits were done in the mid 1960's using state of the art IBM 360 mainframe computers. The 360 was capable of doing up to 16.6 million instructions per second. The larger 360 models could have up to 8 MB of main memory, though main memory that big was unusual—a large installation might have as little as 256 KB of main storage, but 512 KB, 768 KB or 1024 KB was more common. They cost about $10,000 a month to rent. Today's microcomputer technology has grown exponentially since then. They are 64 bit multi-core machines, which run from 3 to 5 billion operations per second, and typically have 8 billion bytes of RAM. My 6 year old Intel i-5 3.1 GHz microcomputer solves and computes a 2D 4-body trajectory problems in a few seconds. The IBM 360 computed at a rate of about 1 Megaflops, my PC does 14,000 Megaflops, more than a thousand times faster. The number of transistors has grown by a factor of 10 million.

Historical Perspective:

Calculate Resultant Velocities from Each Burn of the Rocket Engine Stage

1. Given the Saturn IV engine parameters (mass, type of fuel, gas exhaust velocity, thrust control, thrust angle, azimuth angle, and fuel burn rate), for each of the three stages, we will calculate the engine's direction and thrust.

2. Given these parameters, we will calculate the tangential and normal components of vehicle acceleration. From these accelerations, we can then use Newton's Laws to calculate the components of velocity Δv. From this we will get the resultant orbit's altitude and range of travel for the vehicle.

As a check on our calculation, given the vehicle's orbital altitude and the value of the planet's gravitational acceleration, we can calculate the required target velocity to maintain a stable orbit. That is, the orbit then gives us a way of establishing a target for the simulation of the required vehicle's trajectory and velocity for the given altitude. This allows us to tweak the parameters to achieve the mission

Historical, Conceptual, and Strategic Background for Space Flight

- History of the Laws of Astronomical Motion - Observations by the Ancients
- The Scientific Revolution - Observations before the Telescope
- The Scientific Revolution - The Telescope - Galileo's Observational Astronomy
- Pre Newton - Kepler's Laws of Planetary Motion
- History of the Laws of Motion - Newton's Laws and the Invention of Calculus
- Some Important Concepts in Space Trajectories:
- Mission Success: Reliability Assurance
- A New Paradigm: The Apollo Guidance and Navigation System
- The influence of the Apollo Program on the Development of the Integrated Circuit Industry
- Apollo Re-Entry Navigation, Guidance, Control Solutions/Equations
- Orbital Mechanics of Trans-Lunar Injection (TLI) & Free Return Trajectory
- What is the Best Rocket Design Strategy? Lunar-Orbit Rendezvous Summary
- Description of the Seven Rocket Stages Needed to Get to the Moon and Back
- Strategy: Multi-Stage Burns to Lunar-Orbit Rendezvous (LOR)
- Orbit Approximations and Pertubations

Let's Review the Framework for the Analysis and Design of Earth Moon Trajector

History of the Laws of Astronomical Motion - Observations by the Ancients

Man has been investigating the motion of heavenly bodies for millennia. Early man needed to predict th seasons of the year to determine time the for planting of crops. There was also a spiritual fascination with the heavens. The ancients thought that the heavens were the home of God or the gods. The Babalonians were the first to keep written recorda and made careful observations of the stars and planetary orbits. The Greeks were the first to use mathematical models. Since the time of Aristotle it was thought that the earth was the center of the universe and the sun, moon, and planets traveled around it in circular paths called epicycles. This is known as the Geocentric Theory. In 1543, the Polish astronomer Copernicus, based on his planetary observations, hypothesized that the moon and planets travel around the sun.

The Scientific Revolution - Observations before the Telescope

During a period of time loosely known as The Scientific Revolution, when modern science came of age (1550-1700), a few critical thinkers were responsible for transforming our understanding of the cosmos Tyche Brahe devoted his life to making astronomical observations. He built an instrument called a spherical astrolabe of large size and great accuracy, and with funding from the Danish government, had a team of people working on observations. At that time, he made some of the most accurate observations of the moon, planets, and stars. He created detailed mathematical tables that astronomers used for centuries. Brahe also correctly established the positions of 1,000 fixed stars. In 1588, he published his book *Introduction to the New Astronomy*, which included his observations and his system of the world.

The Scientific Revolution - Observations with the Telescope

In 1608 the Dutch lens make Hans Lippershay filed a patent for the telescope. Galileo created a higher power telescope, did detailed astronomical observations which he published, is considered the Father of Observational Astronomy, and is often incorrectly credited as being the inventor of the telescope.

Pre Newton - Kepler's Laws of Planetary Motion

A major conceptual breakthrough was the publishing of Kepler's Laws in 1609.
From very detailed observations by Tyche Brahe, Kepler deduced three Laws of planetary motion:
(1) All planets move about the Sun in elliptical orbits, having the Sun as one of the foci.
(2) A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time.
(3) The Law of Harmonies: For all planets, the orbital period squared is proportional to distance cubed.

He also derived Kepler's Equation relating the parameters of the ellipse:

$$\mathbf{M = E - e \sin(E)},$$

where M is the mean anomaly, E is the eccentric anomaly, and e is the eccentricity. See the last two pages, ***Glossary for AstroDynamic and Keplerian Model***, for a more detailed explanation. This equation can be used to find the orbital elements of an ellipse for a body from observations of it made from the earth. It is a transcendental equation, it has no closed form solution, that is, no formula for E. It can only be solved by trial and error or numerical iteration. See **Section I** for one recursion method. Kepler reasoned that the motion resulted from "a force in the sun" which moved the planets and varied with the inverse square of distance. The radius r, in a planar Kepler orbit or a conic section, is described by 3 parameters: a geometrical constant (p), angle to its major axis, ν, and eccentricity (e), which is a measure of its shape. Position $r = p/(1 - e \cos(\nu))$. Refer to last two pages for more detail.

Kepler's formulation of elliptical orbits was a foundational improvement, but still did not account for al the variation of orbits, particularly in the orbit of the moon, which varied from a simple elliptical path. In the 17[th] century, there were seven other methods of calculation that provided similar accuracy, some of which also used elliptical orbits. One of the perplexing issues in the 17th century, posed by Hooke and Halley, was whether orbits are mathematically perfect, but might be indefinitely complex.

History of the Laws of Motion - Newton's Laws and the Invention of Calculus

Since Johannes Kepler first formulated the laws that describe planetary motion, scientists endeavored to solve for the equation of motion of the planets. In his honor, this problem has been named The Kepler Problem.

In 1687 Newton published his Principia, which provided both the Physics and an accurate description of the trajectories of astronomical bodies, through his three laws of motion and the inverse square Universal Law of Gravitation. A key principle was the force concept and key definitions of mass, inertia, and centripetal force. To be able to solve his equations, Newton invented the mathematical tool of differential calculus, which formulated the rules by which the trajectory paths could then be determined from his laws. Calculus is essentially the mathematical study of rates of change (in the same way that geometry is a study of shape and space, and algebra is a study of functions). Equations for rates of change are called differential equations. the resulting differential equation for Newton's second law for the determination of trajectories can be seen below and in Section I. Calculus provided the rules for solving differential equations, but given the rules for solving the differential equations, the numerical calculation of trajectories is still a very complex mathematical operation.

$$m_i \ddot{\boldsymbol{r}}_i = -G \sum_{j=1, j\neq i}^{n} \frac{m_i m_j}{d_{ji}^3} \boldsymbol{d}_{ji}$$

Where n bodies have a mass m_i, a distance to mass m_j of d_{ji}, and r_i double dot is the acceleration of i.

Newton's 2nd Law Equation for N Body Problem:

Newton's laws are sufficient (other than the precession in the orbit of Mercury) to the accurately determine the trajectories of all astronomical bodies including comets and spacecraft. The key is the ability to account for other perturbing forces, such as the gravity of the sun, planets, and non gravitational forces such as thrust, drag, solar radiation or rotation. This is the N Body Problem

The philosophical foundations of Newton's Law of Gravity remained under dispute for centuries because there was no physical mechanism for contact. How was this “action-at-a-distance” gravitational force produced? It took Einstein's General Theory of Relativity to provide a satisfactory answer. There were also competing vortex theories championed by well respected mathematicians.

The original study of a planet's motion involved just the gravitational influence of the sun. Since it involves only two bodies, it is called a 2 body problem. Newton's Law of Gravity can be solved exactly for this simple 2 body problem and results in an equation for two circles revolving around a common center (center of gravity) resulting in a rotating elliptical orbits. This deviated from motion centered exactly about the sun or the earth and resulted in wobbles in their orbits.

Development of Classical Mechanics After Newton

The moon's motion as affected by the gravitational pull of both the earth and sun is a 3 body (sun, earth, and moon) formulation. The center of mass of the earth-moon system revolve around the sun. Other mathematicians have given algebraic approximate solutions to the Newtonian 3 body orbital equations. Notable contributions to Classical Mechanics were made by Euler, Lagrange, Laplace, D'Alembert, Gauss, Jacobi, and Lamaître. By ignoring the influence of the sun, trajectories can be calculated for rockets about the moon with a 3 body solution. The orbit of the moon can be described by six classical orbital parameters. To land on a particular point on the surface of the moon, we must also consider its period of rotation and its location relative to a launch point from the earth. We can also get approximate solutions by just patching together different parts of elliptical and conic orbits.

Contemporary Solutions

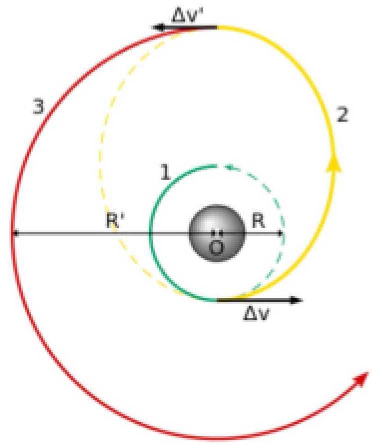
What is the approach we will use to present our simulation and determine trajectories? We will develop our work using Mathcad, a computer computational tool that is specifically made to document and present, enter and manipulate physical and mathematical models and equations, and then plot the results. Using algorithms, computer software programs can solve complex differential equations using numerical methods. Mathcad provides a number of numerical differential equation solvers. Our approach will be to solve Newton's differential equations of motion for the trajectories of both 3 body (earth, moon, and rocket) and 4 body (sun, earth, moon, and rocket) systems using generalized numerical differential equation solvers.

Discussion of important concepts in space trajectories: Hohmann Transfer and Lagrange Point

Before we describe our method, it is useful to introduce two trajectory concepts in Orbital Mechanics: the **Hohmann transfer orbit** and **Lagrange Points**. Below is a description of Hohmann Transfer Orbits from a University of Georgia Math Course.

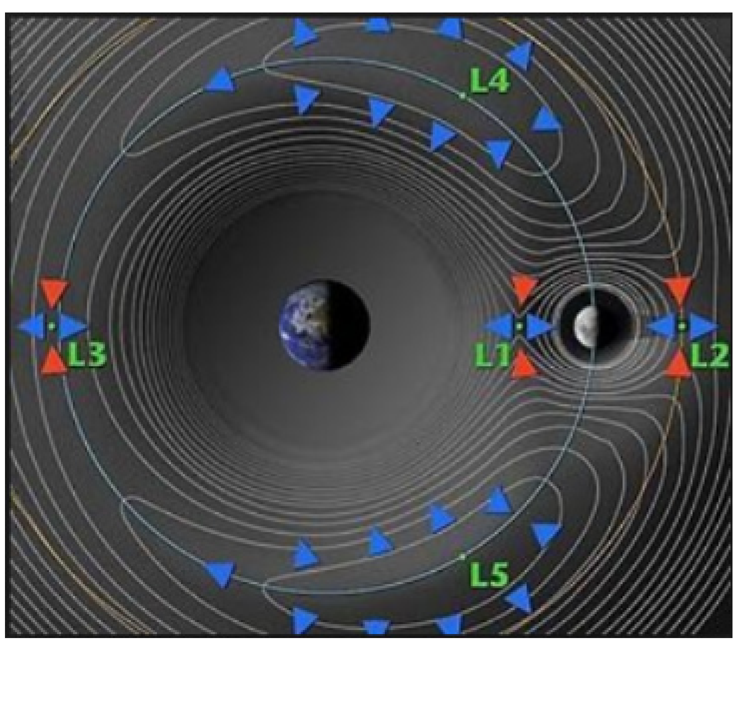
A Hohmann Transfer is an orbital maneuver that transfers a satellite or spacecraft from one circular orbit to another. It is the most fuel efficient way to get from one circular orbit to another circular orbit. Because the Hohmann Transfer is the most fuel efficient way to move a spacecraft, it is a fairly slow process and is used mostly for transferring spacecraft shorter distances.

Hohmann transfer orbit



A Hohmann Transfer is half of an elliptical orbit (2) that touches the circular orbit the spacecraft is currently on (1) and the circular orbit the spacecraft will end up on (3). It takes two accelerations to get the original orbit to the destination orbit. To move from a smaller circular orbit to a larger one the spacecraft will need to speed up to get onto the elliptical orbit at the perigee and speed up again at the apogee to get onto the new circular orbit. To move from a larger circular orbit to a smaller one, the processes are reversed.
http://jwilson.coe.uga.edu/EMAT6680Fa05/Bacon/hohmanntransfers.html

A Lagrange point is a location in space where the combined gravitational forces of two large bodies, such as Earth and the sun or Earth and the moon, equal the centrifugal force felt by a much smaller third body. The interaction of the forces creates **a point of equilibrium** where a spacecraft may be "parked" to make observations. There are five Lagrange points around major bodies such as a planet or a star. Three of them lie along the line connecting the two large bodies. In the Earth-moon system, for example, the first point, **L1**, lies about 5/6th of the distance between the Earth and the moon, at about 1 million miles from Earth. The gravitational pull of the earth balances that of the moon at **L1**. See the illustration of lines of equal gravitational force below.



A New Paradigm: Apollo Integrated Ckt, (IC) Based Guidance Computer (AGC)

Although the main computation was done with mainframes, a computer in the spacecraft was still justified. There was still a 2.5 s time delay in the signal path from earth to the moon and back. Lunar orbit insertion took place at the backside of the moon, blocking any signal path. Trajectory calculations for landing would require an autonomous on board computer. There were also concerns about malicous signal jamming and a desire to prepare for longer planetary trips and possible simultaneous multiple missions. Also, the later decision to use the Lunar Rendezvous method over Direct Ascent proved the wisdom of this decision. Lunar Rendezvous would not be possible without the AGC.

The first contract award by NASA was for the development of the Inertial Guidance System and digital Apollo Guidance Computer (AGC) to the Massachusetts Institute of Technology (MIT) Instrumentation Lab. MIT had developed the navigation systems since the late 1950s for the aerospace programs such as Polaris. At that time the first generation of computers used vacuum tubes and took up a whole room. The AGC was one of the first third generation computer, that is, it used Integrated Circuits for its logic. However, because of the long development cycle of Aerospace programs and the commitment to using only proven parts with reliability established by extensive component testing, environmental testing, component stress de-rating, and qualification, by the time of the Apollo 11 moon flight in 1969, considering its cost and size, it was far less powerful than a DEC PDP-11 (Programmed Data Processor) design, which was in popular use at that time.

The computer was based on digital rather than analog technology to provide sufficient accuracy. The final production version of the AGC used Fairchild Micrologic Resistor-Transistor-Logic (RTL) Integrated Circuits, consisting of three input NOR gates, packaged into flatpaks. This well established NOR gate was the basic building block of the AGC. Fairchild and Texas Instruments shared the patent rights for the invention of the Integrated Circuit. At one time, the AGC used 60% of all the ICs made in the US. The flatpaks were assembled into modules called "Logic Sticks".

The AGC4 used four 16 bit registers, a 15-bit wordlength + 1-bit parity, 2048 word RAM (magnetic-core memory), 38,912 words of bank switched ROM (high density, low power, magnetic core rope memory), 5600 logic gates, and ran at a basic machine cycle of 2.048 MHz. It had an add cycle time of 23.4 µs and double precision add and multiply subroutine times of 235 and 780 µs.

It used 34 instructions for 45 different programs with 80 different "verbs" and 90 "nouns." The AGC had a real-time, multiprogramming, priority scheduled, interrupt driven, operating system. It had logic alarms and employed double word capability for accuracy.

It had 8 I/O ports (Inertial Measurement Unit, Display, 2 radar, hand controller, telemetry, engine command, and RCS). It occupied 2 cubic feet, was hermetically sealing to prevent corrosion, weighed 70 lbs, and consumed 50W of power. It had a keyboard for data input and used a digital electroluminescent display (DSKY) to provide the human interface. The astronauts said that they preferred analog meters for readout, but the digital display provided a far broader range of information.

The power was generated from hydrogen/oxygen fuel cells. There were three fuel cells whose output was 27 to 31 volts. Normal power output for each power plant is 563 to 1420 watts, with a maximum of 2300 watts. Each cell weighed about 200 lbs. As a side benefit, the fuel cells generated about 20 gallons of water per day, used as drinking water by the crew.

The influence of the Apollo Program on the Development of IC Industry & Information Tech
50 Years of Microprocessor Development and the Death of Moore's Law

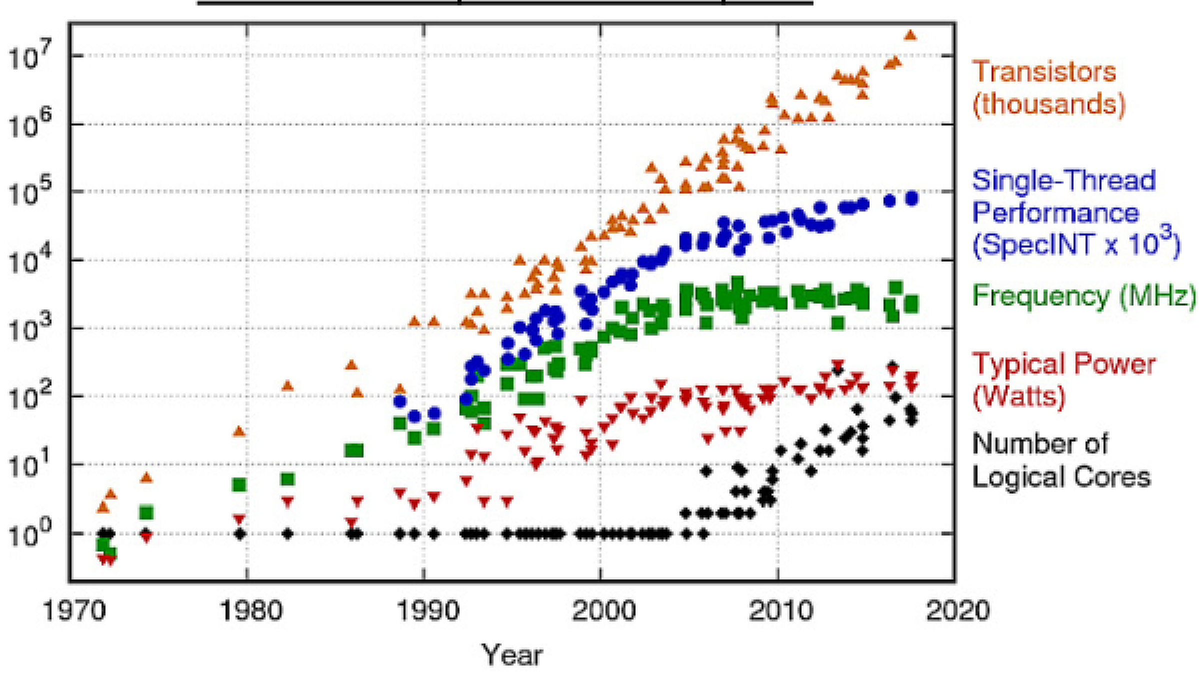
The development of NASA's Apollo technology has changed history. The Apollo lunar program made a staggering contribution to various aspects of high tech development, especially ICs and computers. It costs about $10,000 a pound to launch a vehicle to lower Earth orbit. This put great value on ICs which were far more compact, weighed less, and used less power that the discrete transistors used on earlier space computers, such as for the Polaris. And, as noted above, they required fewer connections and thus were more reliable.

However, in the early 1960s they ICs were far more expensive, costing about $1,000 each & were only available in limited quantities. In the summer of 1963, 60% of the total US output of ICs were being used in Apollo prototype construction. Product development improvements in the integrated circuit industry are based on the concept of the learning curve. The learning curve relates reduction in manufacturing cost to cumulative production volume. One form of this is **Moore's Law**. In 1965 Gordon Moore of Intel stated that complexity for minimum component cost will double/yr. In 1975 he revised it to state that there is a doubling of the number of transistors in an IC every 2 years. By 2012, Intel had changed this to every 2 1/2 years. From the curve below it can be seen that improvements in clock rate, power device, and single thread performance stopped in 2003. Traditional Dennard scaling based on reduced voltage and feature size came to an end. The number of transistors continued to increase by using new types of connect and gate materials and the introduction of the FinFet transistor.

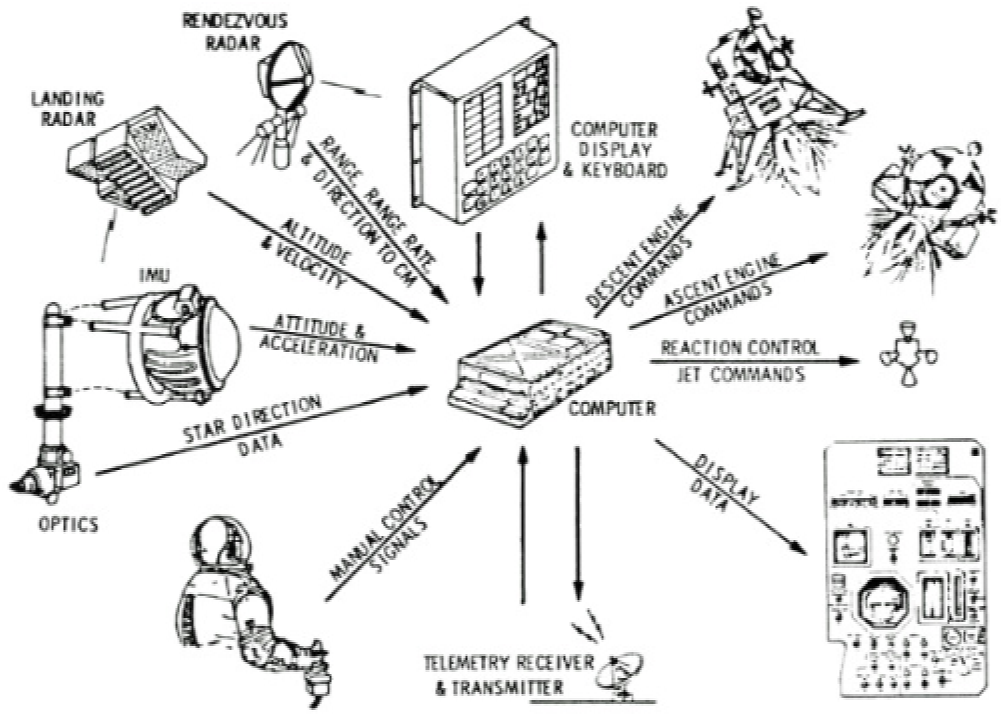
It was this early Apollo purchase of a volume of very expensive ICs that helped jump start the integrated circuit and microcomputer industry into further development, production, and marketing. Military purchases initiated the beginning of Moore's Law in 1965. In 1970, Intel introduced the first microprocessor (µP), the Intel 4044. The growth of Moore's Law is now measured in terms of µP transistor count and performance improvements. The increased transistor count comes by adding more processing cores for types of applications that can make use of multiple threads or parallelism.

The impact of IC growth on our life is hard to overestimate. From computers to smart phones, the Internet, to TVs, the growth of electronics technology, fueled by advances in ICs, has been phenomenal. The impact of these developments has been so profound that it is now often taken for granted: consumers have come to expect increasingly sophisticated electronics products at ever lower prices, such as new Apple iPhone models every year or so. It has been predicted because of both physical limits and increased development and fabrication costs that Moore's Law will come to an end in the 2020s. It is expected that this will have a profound impact the world's continuing technical innovation and the growth of the world's economy. However, GPU performance and memory storage continue to increase.

50 Years of Microprocessor Development



Apollo Guidance and Navigation System



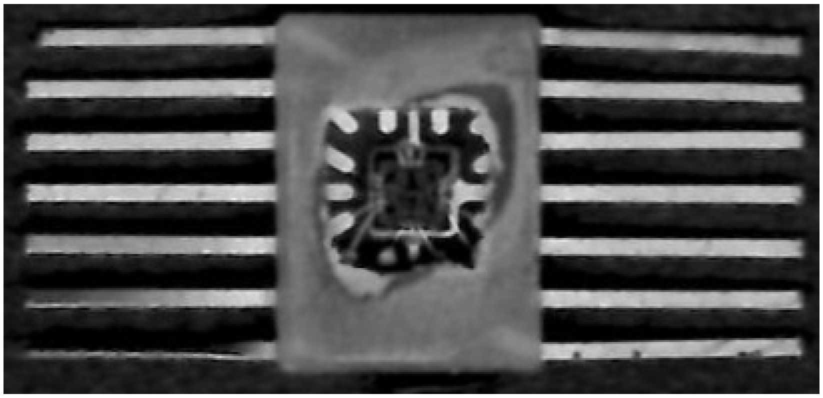
Mission Success: Reliability Assurance

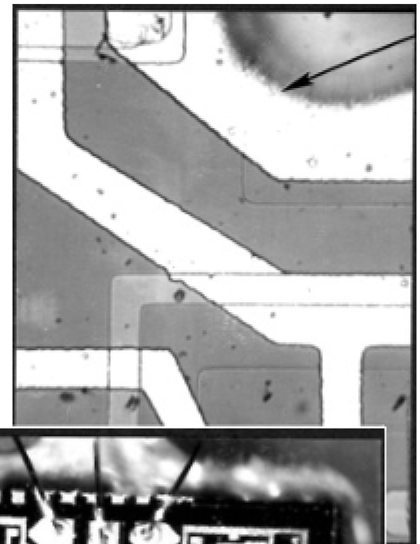
Despite the fact that Russia was ahead of the US in space technology in the early 1960's, the US was the first to get a man on the moon. One of the reasons the US was able to do this was our understanding of the importance of reliability. The computer had to endure the vehicle launch, vibration, radiation, extreme temperatures, and vacuum of space. We incorporated reliability techniques of failure mode and effect analysis, the component failure analysis and closed failure reporting and corrective action scheme, the use of proven parts and techniques, the pursuit of simplicity, component de-rating, the elaborate deployment of redundancies, individual component qualification, component lot qualifications, testing of components, environmental testing of components and subsystems under simulated environments: such as vacuum, vibration, and extreme temperature, and the systematic implementation of design reviews.

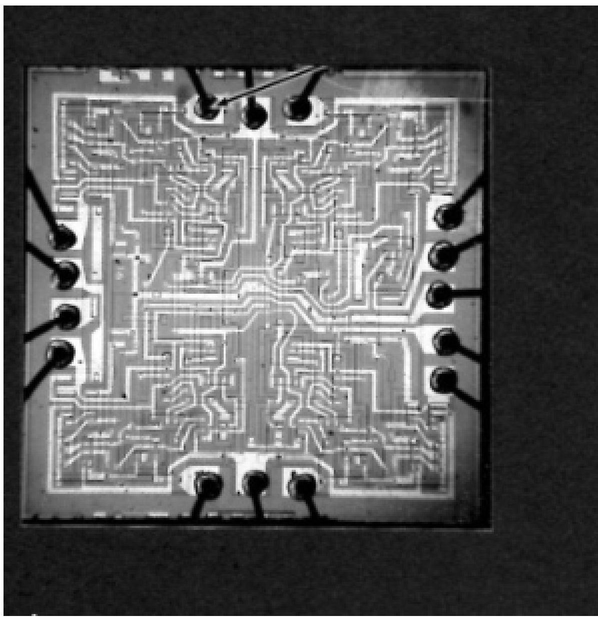
Probabilistic Reliability Program: Functional diagrams representing the relationships between these components and subsystems and component test data are then translated into statistical failure rate and reliability terms using Statistical Models to insure that Reliability goals are achievable with the chosen design approach.

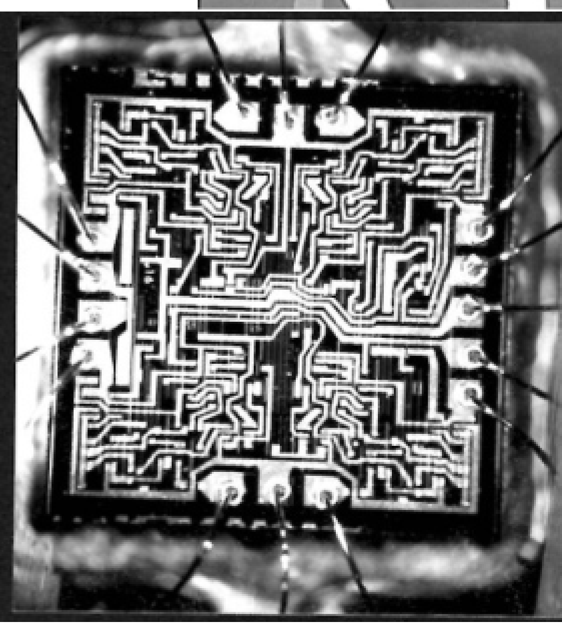
The AGC Reliability Requirement was a mission success probability of 98.6%. Let's examine three specific failure modes exhibited by the AGC ICs: 1). open bonds caused by a gold rich, aluminum-gold intermetallic between the gold wires and the aluminum bond pads called "purple plague", 2). shorting caused by loose conducting particles, and 3). electrical leakage of the isolation region of bond pads cause by a defective isolation diffusion masking operation. The photomicrographs on the following page show an example on an Aerospace flatpak Logic IC with a leaky bond pad on which I did a failure analysis back in the late 1960s. The small arrows point to the leaky bond pad and the area of the small mask defect.

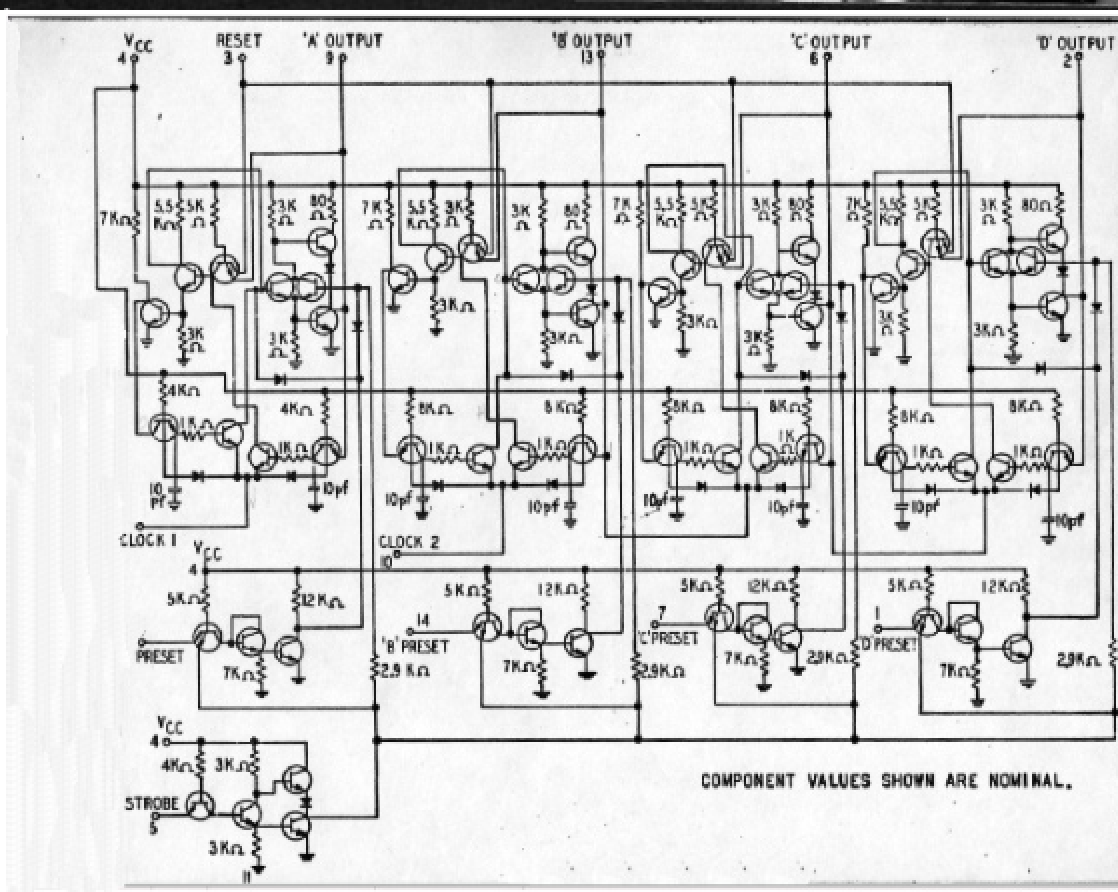
Failure Analysis of RTL Micrologic IC Reveals Mask Defect in Die as Cause of Failure











Apollo Re-Entry Navigation, Guidance, Control Solutions/Equations

The whole purpose of a Navigation, Guidance and Control System is to answer three questions:

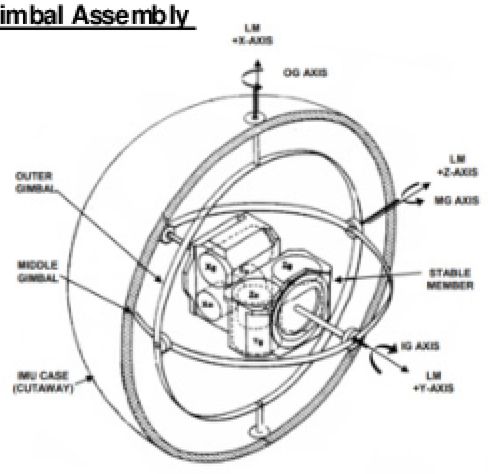
1. Where am I?
2. Where am I going?
3. What do I have to do to get there?

1. Where am I?

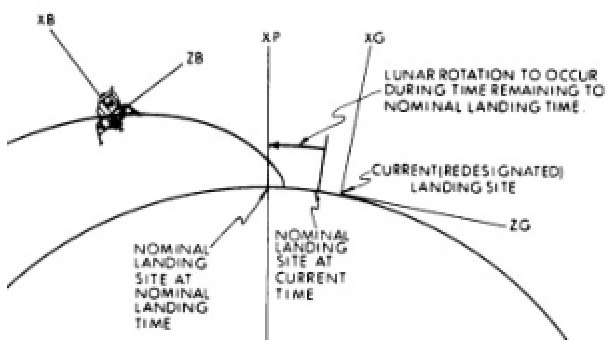
The **Inertial Measurement Unit** (IMU) is a three-degree-of freedom gimbaled platform isolating **three single-degree-of-freedom gyros** and three single-axis accelerometers from the spacecraft attitude angle. Its purpose is to provide a stable platform for measurement of attitude and acceleration and to provide isolation from its case by three orthogonal (x, y, z) gimbals. Accelerometers are small mechanical devices that respond to accelerations of the vehicle. Each accelerometer measures acceleration in a single direction; therefore the three accelerometers are used to take measurements of the complete motion of the vehicle in space.

The orientation of the platform and the direction of the sensitive axes of the accelerometers are held inertially, fixed by the gyro error signals, which feed the platform drive servos. The orientation is held to a fixed position. The attitude is then determined by alignment to the stars. The IMU provides the computer with information of spacecraft attitude by readout of the IMU gimbal as shown in the illustration below:

IMU Gimbal Assembly



Coordinate Frames:
IMU Platform, Guidance Coordinates
LM Body Spacecraft Coordinates



2. Where am I going?

Apollo Guidance and Control Solutions, and Rendezvous Equations

Targeting is the process of determining the target coordinates on a rotating earth, and the location and rotation of the vehicle with respect to these coordinates. This information is mandatory to the operation of the closed-loop guidance system and is calculated at every pass through the guidance logic. At a pre-selected value of velocity, the targeting will switch to relative coordinates. The control system controlled thrust and vehicle attitude.

3. What do I have to do to get there?

Trajectory Model: Quartic Polynomial, with 4 spatial derivatives of motion
Reference: *Apollo Lunar-Descent-Guidance*, MIT, June 1971

The parameters which are calculated in this phase are: (1) the predicted target vector, which accounted for a predicted earth rotation based on an estimated flight time, (2) the range-to-go, which is simply the arc cosine or the vector dot product of the position vector (time derivative of the position) of the vehicle and the predicted target vector, and (3) the lateral angle, which represented the angle between the target vector and a radius vector formed by the intersection of the calculating plane and a plane which contains the target vector and its perpendicular to the calculating plane. The angle is used in the lateral logic to control the direction of the roll command. It is convenient to think of the reference trajectory as evolving backwards in time from the target point, with the time variable T reaching zero at the target point and negative prior to that point. The situation is dynamic, the target point keeps changing.

Thus the target-referenced time (T) is to be distinguished from clock-time (t). Because guidance gains would become unbounded, the target point is never reached. Instead, a guided phase is terminated at a negative time T and the succeeding phase is started. Both the terminus and the target point lie on the reference trajectory, but the target point lies beyond the portion that is actually flown. In terms of a vector polynomial function of target-referenced time, we wish to define a reference trajectory that satisfies a two-point boundary value problem with a total of **five degrees of freedom** for each of the **3 space dimensions.** This number of degrees of freedom is required in order to constrain terminal thrust in the Break Phase Program (P63) and to shape the trajectory design in the Approach Phase Program (P64) for targeting. **A quartic polynomial** is the minimum order with which five constraints on the reference trajectory can be satisfied. With the reference trajectory evolving backwards in time from the target point, it can be defined as:

Control Solution Rendezvous Equation:

$$RG = R\_RTG + VTG \cdot T + \frac{ATG \cdot T^2}{2} + \frac{JTG \cdot T^3}{6} + \frac{STG \cdot T^4}{24}$$

where
RG is the current position vector on the reference trajectory in guidance coordinates at the negative time T.
RTG is the target position, and
VTG, ATG, JTG, and STG are the four target position time derivatives:
velocity, acceleration, jerk, and snap, respectively.

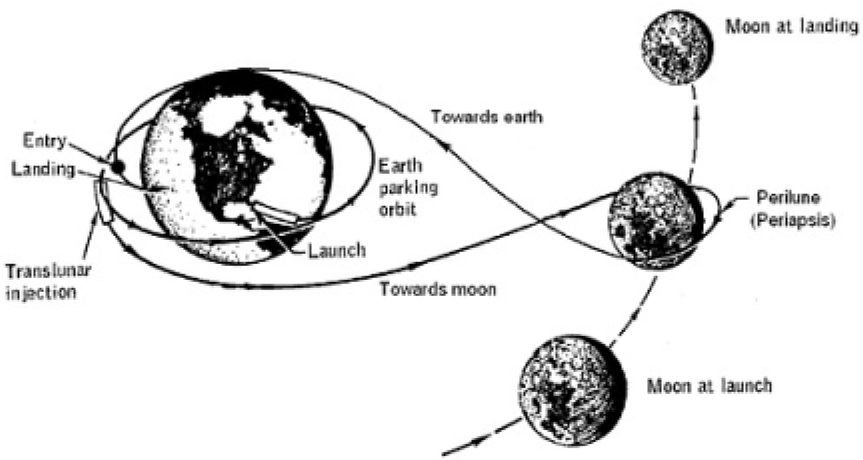
Saturn V Flight Control

The S-IC stage was powered by five Rocketdyne F-1 engines arrayed in a quincunx (five units, with four arranged in a square, and the fifth in the center). The center engine was held in a fixed position, while the four outer engines could be hydraulically turned (gimbaled) to steer the rocket. The final velocity is determined by the rocket's weight, thrust, burn time, attitude, and for the Service Module, pulse rate. IBM supplied the guidance computer. It occupied a 3-foot-high section of the 360-foot-long rocket, sitting on top of the third stage.

Orbital Mechanics of Trans-Lunar Injection (TLI) & Free Return Trajectory

The Trans-Lunar Injection (TLI) was a maneuver used to change the trajectory of the spacecraft from the circular earth parking orbit to a highly eccentric elliptical orbit. An ellipse is an oval shape. It thus has two radii of different lengths. The longer radius is called the apogee. The shorter is the perigee.

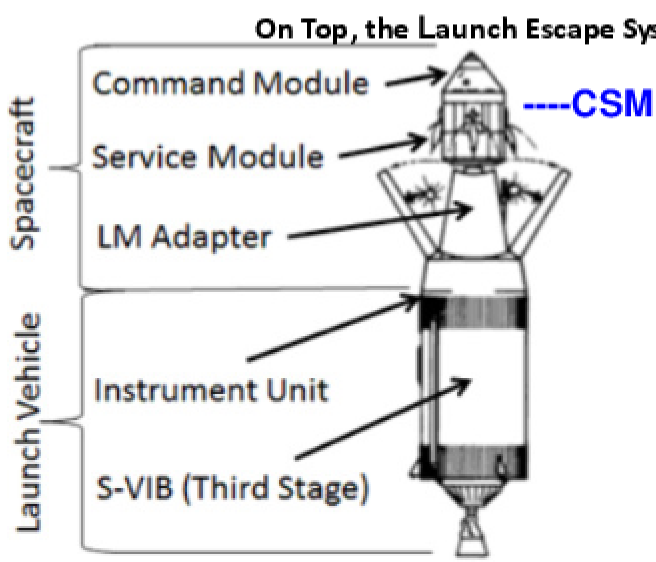
The end of the Stage 3 burn (3B) moves the spacecraft to the earth orbit apogee (farthest point), which coincides with the radius of the moon's orbit. The burn was then critically timed so that the spacecraft neared its orbital elliptical apogee as it approached the moon. The spacecraft was then captured by the moon's gravity and then swung around into a figure 8 orbit around the moon. Apollo 8 did not land. It was carried back to earth in a Free Return Trajectory. Apollo 11 did a retro burn to slow down into an orbit around the moon.



The Basic Idea

Start with a parking orbit around the earth, then do a TLI rocket burn to increase the earth apogee (farthest) orbit of the earth to intersect the apogee of the moon. Circularize the moon orbit, then do a retro burn to descend to the moon.

Apollo TLI Spacecraft - Command/Service Module (CSM)



Orbital Mechanics of the Free Return Trajectory (FRT) - See Section I Below

In Section I, we solve for the 3 and 4 body Gravitational Equations of Motion for the *Free Return Trajectory (FRT)*; that is, an orbit such that a Trans-Lunar or Lunar Orbit rocket burn is not required for return to earth. It uses the pull of the moon's and Earth's gravity. This provides for a way to allow a return without the need for propulsion, in the event of a mission failure. This method was an option for Apollo 8, Apollo 10, and Apollo 11 missions and was the emergency method used for return in the aborted Apollo 13 lunar mission. For a lunar landing, a mid-course correction was then made to change from the Free Return Trajectory going to earth to a Trans-Lunar Injection for lunar orbiting and descent.

For simplicity, we will consider only the gravity of the earth, moon, and CSM and ignore the pull of the sun on the CSM. Now the earth moves in its equatorial plane, while the moon's orbit is inclined to the Equatorial plane, between about 18 to 29 degrees. The moon's orbital plane crosses only twice a month. We will consider only a 3 body planar single point mass type FRT; that is, moon is in the same plane as the earth.

What was the Best Vehicle Strategy to Get to the Moon and Back?

From NASA Fact Sheet: https://www.nasa.gov/centers/langley/news/factsheets/Rendezvous.html

NASA gave serious consideration to three vehicle strategies: Initially, direct ascent; then, Earth-orbit rendezvous (EOR), and, finally, a dark horse candidate, lunar-orbit rendezvous (LOR).

Direct ascent. This was basically the method that had been pictured in science fiction novels and Hollywood movies. A massive rocket the size of a battleship would be fired directly to the moon, land and then blast off for home directly from the lunar surface. The trip would be like that of a chartered bus, moving from point A to point B and back to A again in one brute of a vehicle.

NASA realized that any single big rocket had to carry and lift all the fuel necessary for leaving the Earth's gravity, braking against the moon's gravity as well as leaving it, and braking back down into the Earth's gravity again. This clearly was not a realistic option, especially if the mission was to be accomplished anywhere close to President Kennedy's timetable. The development time of a rocket that enormous (about twice the thrust of Saturn V) would be far too long, and the expense would be enormous.

Earth-orbit rendezvous (EOR). The main idea of EOR was to **launch two spacecraft** into space, independently, using two independent Saturn rockets. This would mean that there would be a rendezvous and docking in Earth orbit, followed by assembly, fueling, and detaching of a lunar mission vehicle from the joined modules. This bolstered ship would then proceed in direct flight mode to the moon, followed by direct flight back to Earth orbit. Advantage: **much less weight.**

Lunar-orbit rendezvous (LOR) - dark horse: Injection Trajectory to Moon (See trajectory illustration on previous page)
The basic premise of LOR was to fire an assembly of three spacecraft into Earth orbit on top of a single powerful (three-stage) rocket. The lunar orbit rendezvous concept that, in the opinion of many historians, was chief among the reasons why the U.S., in less than a decade, managed humankind's first extraterrestrial excursions. See Graph of Saturn V Weight Loss during Flight in Section XIII.

This assembly included: One, a mother ship, or command module; two, a service module containing the fuel cells, attitude control system and main propulsion system; and three, a small lunar lander or excursion module. Once in Earth orbit, the last stage of the rocket would fire, boosting the Apollo spacecraft with its crew of three men into its flight **Trans-Lunar Injection** (TLI) trajectory to the moon. Reaching lunar orbit, two of the crew members would don space suits and climb into the Lunar Excursion Module (LEM), detach it from the mother ship, and take it down to the lunar surface. The third crew member would remain in the command module, maintaining a lonely vigil in lunar orbit. If all went well, the top half of the LEM would rocket back up, using the ascent engine provided, and re-dock with the command module. The lander would then be discarded into the vast darkness of space or crashed onto the moon (as was done in later Apollo missions for seismic experiments), and the three astronauts in their command ship would head for home.

When Langley engineers first suggested the concept of lunar-orbit rendezvous, NASA had **rejected it out of hand for being too complicated and risky**. However, LOR enjoyed **several advantages over** the other two options. It required less fuel, only half the payload, and less brand new technology than the other methods; it did not require the monstrous Nova rocket; and it called for only one launch from Earth whereas EOR required two. Only the small, lightweight lunar module, not the entire spacecraft, would have to land on the moon. **This was perhaps LOR's major advantage**. Because the lander was to be discarded after use and would not need return to Earth, NASA could tailor the design of the LEM for maneuvering flight in the lunar environment and for a soft lunar landing.

NASA also incorporated a Free Return Trajectory (FRT) option into their flight plans. This is a trajectory where if a lunar landing is aborted (such as the Apollo 13 mission), the initial FRT programmed into the TLI, can then safely return the spacecraft to earth orbit without any additional propulsive engine burns.

Description of the Seven Saturn V Rocket Stages Needed to Get to the Moon

The rocket that carried astronauts to the moon was the Saturn V (referred to as the Saturn Five). It was 363 feet tall, weighed 6.2 million pounds, and consisted of three stages. Stage 1 burned 3 tons of fuel each second. Each stage played a different role in launching Apollo on a path to the moon. The vehicle design strategy used a combination of components of different rockets, and it is called "Lunar Orbit Rendezvous." The components of the ship were discarded, one by one, and then the remaining vessel became the rocket that sent the astronauts into space.

The actual Apollo lunar spacecraft was stacked on top of the Saturn V. It was made up of three parts: 1) the lunar excursion module (LEM), the component that would eventually land on the lunar surface; 2) The service module (SM) which had propulsion systems for course corrections and entering and escaping the moon's and earth's orbits; and 3) the command module (CM), the compartment occupied by the three astronauts for most of the mission. Last but not least, is the stage located at the very top, the Launch Abort/Escape Rocket System which was designed to pull the command module away from the rocket in the event of problems during launch. Together, all these pieces made up the Saturn V rocket and Apollo 11 spacecraft.

It's the way they were functionally connected and then jettisoned, that made the moon landing happen. The Saturn V's first stage launched the Apollo spacecraft, which carried the craft 42 miles above the earth and reached a speed of about 6,000 miles per hour. The first Saturn V stage then detached. Once the Saturn V second stage kicked in, the now needless launch escape system was jettisoned, and the second stage propelled the spacecraft even farther and faster into space.

After second stage detached, the third stage of the rocket fired briefly to knock Apollo into a **parking orbit**, 103 miles above the Earth's surface. Here final checks were made and the Saturn V third stage fired again to set Apollo on its course to the moon, in a move called the Trans-Lunar Injection (TLI). Once the spacecraft propelled itself away from Earth, the Saturn V's job was done. The astronauts then performed a mid-flight maneuver to reconfigure the ship. The crew could now access the lunar module which had been stored in a protective compartment during launch. To do this, the command service module detached, flipped 180 degrees, and docked with the lunar module. After this maneuver, they jettisoned the last and now useless third stage of the Saturn V rocket.

This whole high-stakes launch process only took about three and a half hours. Thus the payload were the combined Apollo CM, SM, and LEM spacecraft (**CSM**). For the next **77 hours** or 3.2 days

for the Time of Flight (TOF), Apollo coasted through space in the TLI orbit until it finally reached its target. See Section I. It was pulled into orbit by the moon's gravity. This is where the crew split up. Armstrong and Aldrin transferred to the lunar module named Eagle and slowly descended toward the lunar surface, while Collins continued to circle the moon in the command module called Columbia. Now here comes another tricky part: landing on the moon. To make this historic moment happen, Eagle turned around and used its engine to slow its speed, and ultimately touched down on the lunar surface.

The second phase occurred after about 21 ½ hours later. After the moon walk, Eagle performed the first vehicular launch from a celestial body that wasn't the Earth. Then, leaving its landing gear behind, timing its ascent with Columbia's path in lunar orbit, it rejoined the CM spacecraft. Once Collins and Aldrins transferred back into the command module, the LEM lunar module was no longer needed and it was left behind on the moon. Just like before, Apollo needed to break out of orbit. This maneuver was called the Trans-Earth Injection. It began the two 1/2 day journey home. Upon approaching its entry point into Earth's atmosphere and no longer needing its propulsion engines, Apollo jettisoned the service module and prepared for re-entry. Protected by the now exposed heat shield on the bottom of the command module, Apollo blazed across the heavens, coming back to earth at 25,000 miles per hour. After being slowed by the earth's atmospheric drag, the parachutes deployed, and Columbia splashed down safely into the Pacific Ocean. What started out as a 3,000 ton behemoth of rocket fuel and freight, was reduced to a small command module floating in the ocean, carrying three astronauts and rock samples collected from the surface of the Moon.

Strategy: Multi-Stage Burns to Lunar-Orbit Rendezvous (LOR)

Gravity Turns - Minimum Energy Orbit

Motion in an orbit is circular. Orbital launch requires that the flight end with a roughly horizontal or tangential velocity at orbital speed. One useful maneuver to accomplish this transition is called the gravity turn. In this maneuver, the earth's gravity acts to turn the trajectory of the rocket towards the horizontal. If the attraction from the Moon was not a factor and the purpose of the mission was for Saturn V to reach the escape velocity, the rocket could have blasted off in a vertical direction from the Earth or simply accelerated along the tangential direction or motion. Going in a vertical direction would require the rocket to accelerate from zero velocity to the escape velocity. However, going up to the escape velocity in the radial direction required much less energy.

First Stage - Saturn Stage S-IC

When Saturn V blasted off from the Earth, the first stage burned for 2.5 minutes, lifting the rocket to an altitude of 68 km (**42 miles**) and a speed of **2.76 km/s** (6,164 mph) into an initial Earth-orbit of 114 by 116 miles. This orbital speed was much less than the escape velocity.

Second Stage - Saturn Stage S-II

After the S-IC stage separated from the Saturn V rocket, the S-II second stage burned for 6 minutes. This propelled the rocket to an altitude of 176 km (**109 miles**) and a speed of **6.995km/s** (25,182km/h or 15,647mph). This speed is close to the orbital velocity for that altitude.

Third Stage - Saturn Stage S-IVB Burn #1 - Earth Parking Orbit

After the S-IVB stage separated from the rocket, the third stage burned for about 2.5 minutes. It then cut off, and the Apollo 11 went into a "**parking orbit**" at an altitude of 191.2 km (**118.8 miles**). Its velocity was **7.791 km/s** (28,048 km/h or 17,432mph). It made several orbits around earth.

Third Stage - Saturn Stage S-IVB Burn #2 - Trans-Lunar Injection of CSM

After several orbits, the rocket's engines re-ignited, and it blasted off for what they call ***Trans-Lunar Injection***. According to NASA, Saturn V reached an altitude of 334.436 km (**208 miles**) and an escape velocity of **10.423 km/s**, at which time the engines were shut down. This velocity was less than the escape velocity for that altitude, but it was sufficient to take Apollo 11 to the Moon. The gravitational attraction from the Moon facilitated its motion.

Two Mid-Course Trajectory Corrections by the Service Module (SM) SPS Engine

Free-Return-Trajectory, a Contingency Option - See Section I

This is a special kind of Trans-Lunar-Injection trajectory (ballistic) which, after only the initial first burn, would allow Apollo to flyby the moon and return to earth without any additional burns using only the gravity of moon and earth (the sun also has a large gravitational influence). This trajectory is a contingency option, should any problems develop with the Mission, such as the Apollo 13

Orbit the Moon, and Descent to Surface of the Moon

After 77 hours attain lunar oribt, after 101 hours, 36 minutes, when the LM was behind the moon on its 13th orbit, the Lunar Excursion Module descent engine fired for 30 seconds to provide retrograde thrust and commence descent orbit insertion, changing to an orbit of 9 by 67 miles, on a trajectory that was virtually identical to that flown by Apollo10. An orbit is easier to target than a fixed spot on moon.

Ascent from the Moon to Moon Orbit

Trans-Earth Orbit and Mid-Course Correction by SM SPS Engine

Re-Entry: Atmospheric Braking with Command Module

Just before re-entry the Service Module is jettisoned leaving only the Command Module, CM.

Deploy Parachutes - CM Splash Down After 8 Days

Outline of the Stages of the Flight Analysis and Simulation:

Sections I to XXVI

This is the analytic part of this paper in which we work out trajectories from first principles and analyze the different rocket burns/stages of the mission. Section I starts out with a historical treatment in which we use Kepler's Laws and his Equation to find an approximate trajectory based on two gravitational bodies: First, the earth and the spacecraft and then the moon and the spacecraft. Each divides the space and considers only the nearest sphere of influence, that is, either the earth or moon.

Analysis: Trajectory Determinations
Sections IA, IB, and IC use Newton's Laws to develop more accurate 3 Body (earth, moon, satelilite) and then 4 Body (sun, earth, moon, satelilite) models for trajectories from first principles. We also solve for trajectories using Newton's fundamental approach in Sections XV - Earth to Moon Trajectory and XXV - Moon to Earth Splashdown.

Simulations: Rocket Burns/Stages
Except as noted above, from Section II and onward, we consider only the particular rocket burns and portions of trajectories, associated with one of the spacecraft engine stages, from the earth to the moon, and then back to earth splashdown.

<u>This Analysis Uses Mathcad 14.0 Software</u>

**to Document this Presentation, Solve Equations,
Perform Computations, and Plot the Results.
Mathcad 14 *.xmcd source files are available:
http://www.VXPhysics.com/Space%20Program/**

Outline of the Analysis & Simulation of Stages

Finding the Earth-Moon Free Return Trajectory - a Contingency Option

I. Simple Lunar Trajectories: Kepler's Elliptical Orbits and The Patched Conic Model

Newton's 2nd Law: Simulation of 3 and 4-Body Free-Return-Trajectory

IA. Trajectory Model: 3 Body, Earth, Moon, Spacecraft Planar Point Mass, Earth Center
IB. Trajectory Model: 4 Body, Sun, Earth, Moon, Spacecraft Planar Point Mass, **Earth Center**
IC. Trajectory Model: 4 Body, Sun, Earth, Moon, Spacecraft Planar Point Mass, **Sun Center**
1. Moon's Gravity Turned Off -Trajectory does not return to earth
2. Moon's Gravity Turned On -Trajectory returns to earth

The Saturn IV Rocket - Three Stage Burns to the Moon

II. Table of Saturn IV Engine Parameters
III. Create a Model for Vehicle Pitch Angle
IV. Simulate Atmospheric Drag
V. Stage 1, S-IC - : Simulation Equations for Pitch, Acceleration, Velocity, and Distance
VI. Graph Velocity, Vertical thrust, Tup. Check that Tup > 1 g as Pitch is reduced
VII. Calculate Stage 1 Altitude and Range
VIII. Stage 2, S-II Burn: Velocity Calculation - Assuming 20^o Ascent Angle & ~ 0.35g
IX. Calculate the Required Velocity to go into Orbit at an Altitude of 191 km
X. Stage 3, S-IVB Burn: Earth Parking Orbit Velocity Calculation

Trans-Lunar Insertion - Trip to Moon

XI. Orbital Mechanics: Estimate Velocity Required for Trans-Lunar Injection
XII. Stage 3, S-IVB Burn: Trans-Lunar Injection to the Moon Final Velocity Calculation
XIII. Graph Velocity and Acceleration (gs) Profile of Flight

Command/Service Module (CSM) Trajectory to Moon and Lunar Orbit

XIV. Service Module Engine: Lunar and Earth Orbits & Lunar Module Engine
XV. Trajectory Sim of Apollo Command/Service Module (CSM) from Earth to Moon

Descent to the Surface of the Moon

XVI. Simulation of Descent from Orbit to Moon Surface, Lunar Orbit Descent, LOD
XVII. Simulation of Ascent from Moon Surface to Orbit, Lunar Orbit Ascent, LOA

Trans-Earth Injection and Mid-Course Correction

XVIII. Command Service Module "Columbia" Trans-Earth Injection
XIX. Trans-Earth Coast, Mid-Course Correction, and CM/LEM Separation

Atmospheric Breaking/Drag and Heat Dissipation Considerations

XX. Re-Entry into the Earth's Atmosphere
XXI. Strategies for Dissipation of Heat from Re-entry Atmospheric Braking
XXII. Simulate Drag Force or Drag Coefficient on CM in Five Different Ways
XXIII. Apollo Re- Entry: Velocity, Altitude, & Cd Flight Data versus time
XXIV. Simulation of Atmospheric Braking: Command Module Acceleration and Velocity

Trajectory to Earth, to Moon, and Back

XXV. Trajectory Simulation of Apollo Command Module from Moon to Earth Splashdown
XXVI. Splashdown: Parachute Terminal Velocity

AstroDynamic and Keplerian Model Terms and Definitions

Review: Outline of the Stages of the Flight Analysis and Simulation
Sections I to XXVI

As noted previously, this is the analytic part of this paper in which we work out trajectories from first principles for a "toy" model and analyze the different rocket burns/stages of the mission. Section I starts out with a historical treatment in which we use Kepler's Laws and his Equation to find an approximate trajectory based on two gravitational bodies: First, the earth and the spacecraft and then the moon and the spacecraft. Each divides the space and considers only the nearest sphere of influence, that is, either the earth or moon.

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Sections IA, IB, and IC use Newton's Laws to develop more accurate 3 Body (earth, moon, satellite) from first principles and then 4 Body (sun, earth, moon, satellite) models for trajectories. We also solve for trajectories using Newton's fundamental approach in Sections XV - Earth to Moon Trajectory and XXV - Moon to Earth Splashdown.

Simulations: Rocket Burns/Stages

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Error Analysis: Trajectory Approximations - Perturbations

All celestial bodies of the Solar System follow in first approximation a Kepler orbit around a central body. Deviations from a Kepler orbit are called perturbations. In ancient times, these were a mystery.

There are a number of simplifying approximations that are used in the following work. In the 3 Body Approximation that follows we account for the fact that the moon travels in an orbit around the earth and that the spacecraft travels in an orbit toward the moon. In the 4 Body Approximation we also account for the gravitational influence of the sun and the orbital velocity of the earth around the sun. But there are other motions that we do not consider. In our work, we analyze these motions in only two dimensions, that is, that they move in orbits that are in the same plane. To use a building as an example, we assume they are on the same floor. But a real building has a third dimension: height.

In our work, we assume that the bodies move in the same orbital plane with a fixed velocity or period. The moon is generally not in the same plane as the earth, but the plane of the moon's orbit tilts from above and below by 5.1^o to the earth's orbital plane with a period of 8.8 years. This motion accounts for a solar eclipse when the moon's plane aligns with the solar plane and blocks sunlight. The moon also tilts along its own axis, analogous to the cause of the earth's seasons. There are a number of other variations in the motions and periods of the moon and other astronimical bodies which are also not considered here, such as precession and libration of the moon's orbit.

We also assume that the astronomical bodies are point masses. But the earth and moon are slightly flattened. This causes an error in the pull of gravity close to the earth of about 1%. It has a significant effect on the motion of satellites around the earth. Even the earth's tidal bulge and solar radiation pressure can influence spacecraft over long periods of time. We also ignored the latitude of the launch point from earth. For interplanetary motion, the gravitational effects of the large planets such as Jupiter and Saturn or other nearby astronomical bodies can be significant.

One other very significant factor is calculation errors. Only so many number of decimal places of accuracy are used for calculations. Thus there are cumulative round off errors. Different numerical software methods also have different kinds of errors. Different types of equation solvers give different types of solutions. For example, each has its own unique kind of oscillations or variations.

Another big assumption we implicitly make is that there is a solution to the problem. There are some cases, for example at Lagrange Points, where there may be no solutions. Points where the gravitational forces cancel out. The orbits are potentially unstable.

I. Simple Lunar Trajectories: Kepler's Elliptical Model (Planar Point Mass)

This Section on Kepler is shown for historical interest. Newton's Dynamics is used in all the following Sections

Kepler's E Model (Planar Point Mass 2 Body): **See** the **Glossary** and **Figures** in **last two pages** of this Study

Convert Cartesian Ellipse Eq. in (x,y) to polar (r,ν) coordinates
Ellipse is relative to the **focus**

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 = 1 \qquad r(\nu,e) := \frac{a\cdot\left(1-e^2\right)}{1+e\cdot\cos\left(\nu-\frac{\pi}{2}\right)}$$

$x(a,\theta) := a\cdot\cos(\theta)$ and $y(b,\theta) := b\cdot\sin(\theta)$

$0 \le t < 2\pi$ $\qquad e = \frac{c}{a}$ $\qquad r(x,y) := \sqrt{x^2+y^2}$ and $\theta(x,y) := \operatorname{atan}\left(\frac{y}{x}\right)$

For the moon

$e_m := .0549 \quad d_m := 384400km \quad d_{ap} := 406603km \quad m_m := 7.347\cdot10^{22}kg \quad a_m := \frac{d_{ap}}{1+e} \quad \mu := 3.986\cdot10^5\frac{km^3}{sec^2}$

For the Earth: Mass $m_e := 5.972\cdot10^{24}kg$ $\qquad$ Note: a and b are distances from the center, c

The parameter e is known as the eccentricity. The value of this parameter defines the shape of our orbit. Depending on the value of e there are four kinds of shapes (conic sections), which means there are four kinds of orbits: circle, ellipse, parabola, and hyperbola, for e = 0, < 1, = 1, and > 1, respectively.

$H \equiv 1 \quad e_e := .6 \quad e_c := 0 \quad e_h := 2 \quad e_p := 1.000 \quad \theta := 0, 0.01 .. 2\pi \qquad G\cdot m_e = 3.985\times10^{14}\frac{m^3}{s^2}$

Basics from Newton's Laws: Energy, Momentum, Parameters of Ellipse $\qquad r_o := 300km \quad \phi_o := 0$

$$Energy(v,r) := \frac{v^2}{2}-\frac{\mu}{r} \qquad h\left(v_o,r_o,\phi_o\right) := r_o\cdot v_o\cdot\cos\left(\phi_o\right) \qquad h = r^2\cdot v\nu^2 \qquad h_u(p) := \sqrt{\mu\cdot p}$$

$$p\left(v_o\right) := \frac{h\left(v_o,r_o,\phi_o\right)^2}{\mu} \qquad a\left(v_o\right) := \frac{-\mu}{Energy\left(v_o,r_o\right)} \qquad e_{traj}(v,a) := \sqrt{1-\frac{p(v)}{a}}$$

Period of Moon Sat Orbit

$$T := 2\cdot\pi\cdot\sqrt{\frac{a_m{}^3}{G\cdot m_e}} = 99.98\cdot hr$$

$$r_h(\theta,e) := \frac{H}{1+e\cdot\cos\left(\theta+\frac{\pi}{2}\right)}$$

If we can Solve for Eccentric Anomaly, E, we get Time of Flight, TOF, t - T

$$\cos(\nu) = \frac{p-r}{e\cdot r} \qquad \nu(p,r,e) := \operatorname{acos}\left(\frac{p-r}{e\cdot r}\right)$$

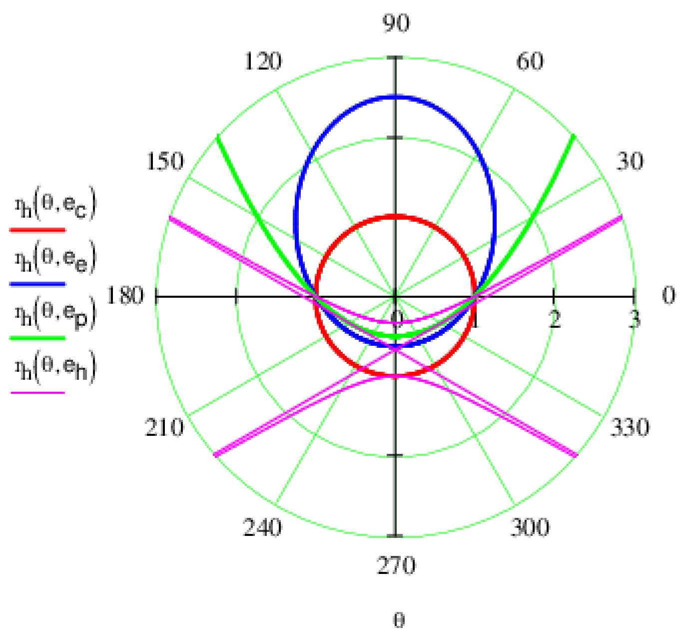
Recursion for Eccentric Anomaly, M & E (Deg)

mean anomaly M (in deg (0<= M<=360)

$$MA\left(M_o,t,t_o\right) := M_o+\sqrt{\frac{\mu}{a_m{}^3}}\cdot\left(t-t_o\right)$$

```
EcA(e,M,dp) := | mx_it ← 30
               | i ← 0
               | K ← π/180
               | del ← 10^(−dp)
               | m ← M/360
               | m ← 2·π(m − floor(m))
               | E ← m  if e < 0.8
               | E ← π  otherwise
               | F ← E − e·sin(m) − m
               | while |F| > del ∧ i < mx_it
               |   | E ← E − F/(1 − e·cos(E))
               |   | F ← E − e·sin(E) − m
               |   | i ← i + 1
               | E ← E/K
```

Plot of Conic Orbits: c, e, p, h



Find E and ϕ In Degrees

t/T: $\frac{27}{360} = 0.075 \qquad EcA\left(e_m,27,5\right) = 28.501 \qquad \phi(e,EcA) := 90-\frac{180\cdot\operatorname{atan2}\left(\sqrt{1-e^2}\cdot\sin(EcA),\cos(EcA)-e\right)}{\pi}$

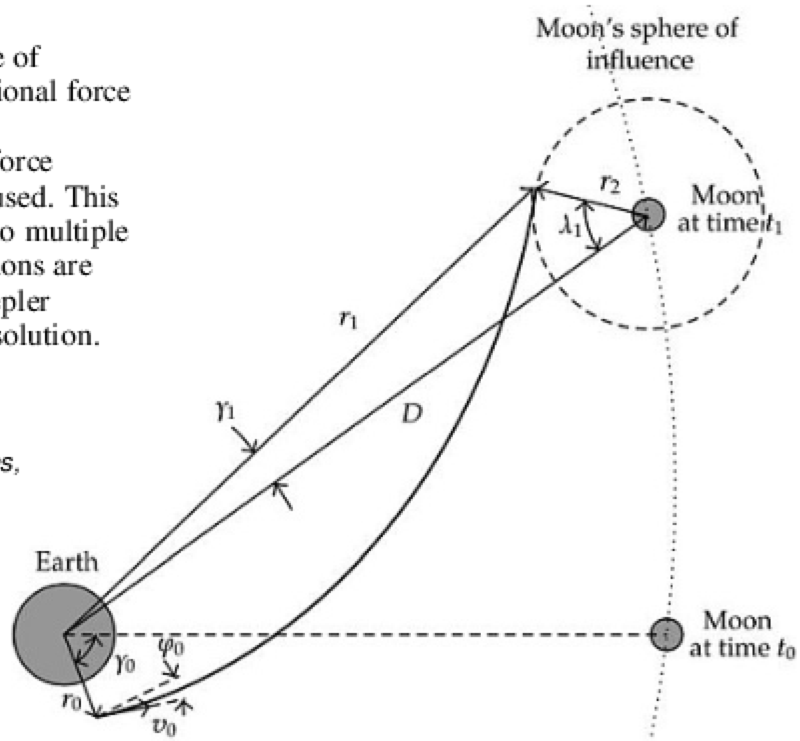
$\phi(0.977, 48.43418\cdot deg) = 153.029$

The Patched Conic Section Approximation for Finding a Lunar Trajectory

The Patched Conic Method is an Approximation for finding a trajectory by dividing space between the sphere of influence (SOI) of the earth, Lunar Earth Orbit (LEO) and the SOI region of the moon.

When the spacecraft is within the sphere of influence of the moon, only the gravitational force between the spacecraft and the moon is considered, otherwise the gravitational force between the spacecraft and the earth is used. This reduces a complicated n-body problem to multiple two-body problems, for which the solutions are the well-known conic sections of the Kepler orbits. Below is an example composite solution.

See for Example:
Optimal Two-Impulse Trajectories with Moderate Flight Time for Earth-Moon Missions,
Sandro da Silva Fernandes
Mathematical Problems in Engineering
Vol. 2012, Article ID 971983,
or
Bate, R. R., D. D. Mueller, and J. E. White,
Fundamentals of Astrodynamics



Rather than dealing with large powers of 10, we can use **Astronomical Units**, for distance, velocity, time: AU, VU, TU. Where AU is the mean distance of the earth to the sun and DU is the radius of the earth. TU is the time unit. Then the velocity unit, (VU) is equal to DU/TU.

$DU := 6378.145km$ $\quad AU := 1.496 \cdot 10^8 km$ $\quad kmps := \frac{km}{s}$ $\quad VU := 7.905368kmps$ $\quad TU := 806.8s$ $\quad D := d_m$

Laplace's Equation for Moon's Sphere of Influence:
this is about 1/6 of the distance, D, to the moon

$R_{sif} := D \cdot \left(\frac{m_m}{m_e}\right)^{0.4}$ $\quad R_s := 66300km$ $\quad R_s = 10.395 \cdot DU$

The conic patched problem for finding a trajectory can be stated as follows:

Given: Initial rocket launch conditions in the earth's sphere of Influence, that is, initial position, velocity, flight path angle, and phase angle: **r_0, v_0 ϕ_0, and γ_0** ,

The three quntities **r_0, v_0 ϕ_0** will give us initial energy and anglular momentum.

Find: Arrival conditions at moon's Sphere of Influence: r_1, v_1 ϕ_1, λ_1. **r_0, v_0 ϕ_0, and λ_1**

The problem with assigning these initial points is that they may not give a satisfactory solution to match the arrival conditions. Our strategy is to use the arrival ange λ_1 to the moon's SOI as one of the independent condition .

Given the 3 initial conditions and one arrival condition as our **independent variables:**
These will move us into the radius of the moon's sphere of influence. Some trial and error may still be required.

EXAMPLE: See Bate, R. R., D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*

Solution: Select the **Apollo 11 Flight Conditions** for initial conditions: $\mathbf{r_0}$, $\mathbf{v_0}$, $\mathbf{\phi_0}$ **and** $\mathbf{\lambda_1}$.

Given: $r_0 := DU + 334km$ $v_0 := 10.6kmps$ $\phi_0 := 0deg$ A reasonable angle to arrive at moon $\lambda_1 := 30deg$

Find: $\mathbf{r_1}$, $\mathbf{v_1}$, $\mathbf{\phi_1}$, $\boldsymbol{\gamma_1}$ (the last symbol, γ, is the Greek letter gamma, the Arrival Phase Angle at the Moon)

Initial Energy and Angular Momentum are $Energy(v_0, r_0) = -0.011 \cdot VU^2$ $h_0 := h(v_0, r_0, \phi_0) = 1.441 \cdot \frac{DU^2}{TU}$

$D = 60.268 \cdot DU$ By the Law of Cosines: $r_1(\lambda_1) := \sqrt{D^2 + R_s{}^2 - 2D \cdot R_s \cdot \cos(\lambda_1)}$ $r1 := r_1(\lambda_1) = 51.529 \cdot DU$

From Law of Conservation of Energy and Momentum: $E_0 := Energy(v_0, r_0)$ $E_0 = -0.011 \cdot \frac{DU^2}{TU^2}$ $h_1 := h_0$

$v_1(r_1) := \sqrt{2 \cdot \left(E_0 + \frac{\mu}{r_1}\right)}$ $v1 := v_1(r1) = 0.128 \cdot VU$ $v1 := 0.1296VU$ $\phi_1 := acos\left(\frac{h_1}{r1 \cdot v1}\right)$ $\phi_1 = 77.542 \cdot deg$

In order to calculate the **Time of Flight**, TOF, to the moon's SOI, we need to Find:

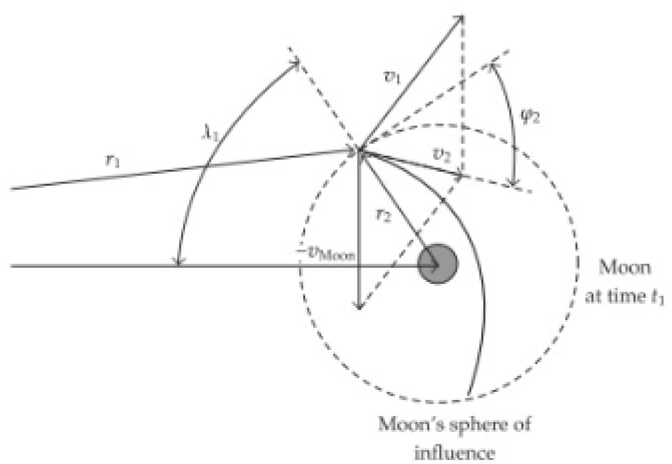
p, a, e, $\mathbf{E_0}$ **and** $\mathbf{E_1}$ for the Geocentric Trajectory.

$p := \frac{h_0{}^2}{\mu} = 2.075 \cdot DU$ $a := \frac{-\mu}{2\,Energy(v_0, r_0)}$ $e := \sqrt{1 - \frac{p}{a}}$ $e = 0.977$ $\nu_1 := \nu(p, r1, e)$ $\nu_1 = 2.956$

$\gamma_1 := asin\left(\frac{R_s}{r1} \sin(\lambda_1)\right) = 5.789 \cdot deg$ $a = 44.698 \cdot DU$ since: $\nu_0 := 0$ $EcA_0 := 0$ $EcA_1 := acos\left(\frac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right)$

$EcA_1 = 1.728$ $TOF := \sqrt{\frac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]$ $TOF = 51.132 \cdot hr$

We can use the same procedure at the moon (Selenocentric).
See Section XVI for the Newtonian Gravitational Solution for the Lunar Trajectory.
We need to determine the values of v1 and Rs in units based on the moon's gravitational attraction parameters.
The Angular Velocity of the Moon (ω_m) in its orbit is



: $\omega_m := 2.649 \cdot 10^{-6} \frac{rad}{s}$ $\omega_m := 2.137 \cdot 10^{-3} \frac{1}{TU}$ $\gamma_0 := \nu_1 - \nu_0 - \gamma_1 - \omega_m \cdot TOF$ $\gamma_0 = 135.637 \cdot deg$

$v_{1m} := 1.024kmps$ $\mu_m := 4093 \frac{km^3}{s^2}$ $v_m := 1.018kmps$ Then $v_{2m} := 1.198kmps$

$\varepsilon_2 := 5.68deg$ $e_m := 2.078$ $r_p := 4105km$ $h_p := 2367km$ $R_s = 10.395 \cdot DU$

$\mu_m := 4.903 \cdot 10^3 \cdot \frac{km^3}{s^2}$

Time of Flight

Develop an algorithm to Calculate Time of Flight

$$TOF_{alg}(v_0, r_0, \phi_0, \lambda_1) := \left| \begin{array}{l} h_0 \leftarrow r_0 \cdot v_0 \cdot \cos(\phi_0) \\ p \leftarrow \dfrac{{h_0}^2}{\mu} \\ E_0 \leftarrow \text{Energy}(v_0, r_0) \\ EcA_0 \leftarrow 0 \\ a \leftarrow \dfrac{-\mu}{2E_0} \\ e \leftarrow \sqrt{1 - \dfrac{p}{a}} \\ r1 \leftarrow \sqrt{D^2 + {R_s}^2 - 2D \cdot R_s \cdot \cos(\lambda_1)} \\ \nu_1 \leftarrow \nu(p, r1, e) \\ EcA_1 \leftarrow \text{acos}\left(\dfrac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right) \\ TOF \leftarrow \dfrac{\sqrt{\dfrac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]}{hr} \\ A \leftarrow (TOF \;\; e)^T \end{array} \right.$$

This gives a different value

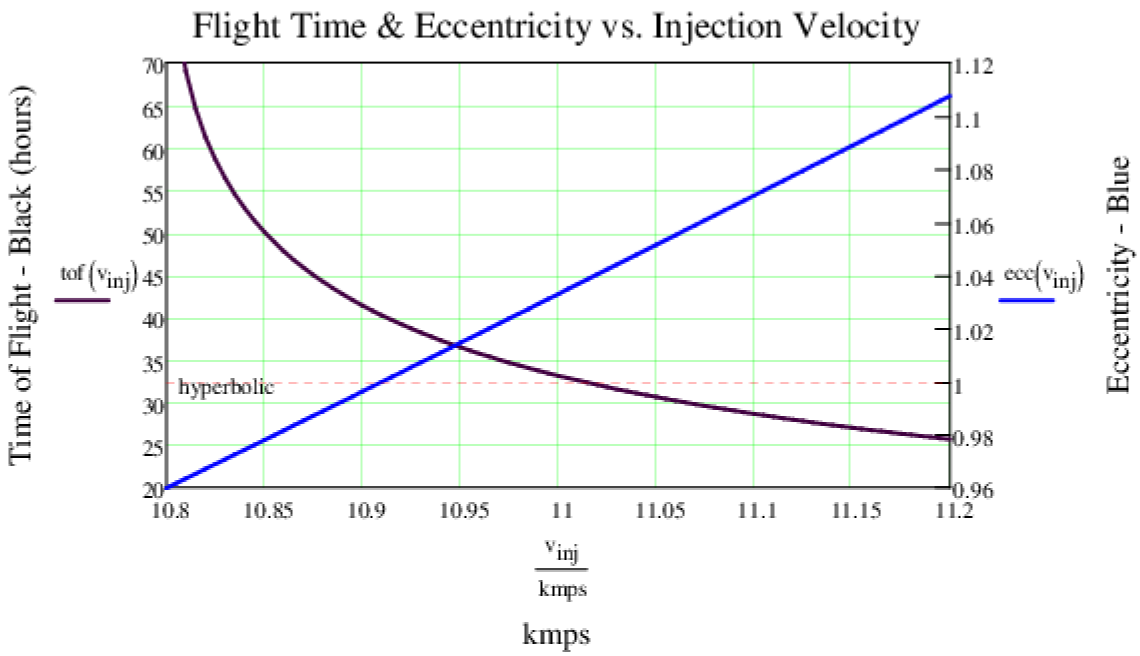
$v_0 = 10.846 \cdot kmps$ $\quad TOF_{alg}(v_0, r_0, \phi_0, \lambda_1) = \begin{pmatrix} 51.132 \\ 0.977 \end{pmatrix}$

$tof(v_0) := TOF_{alg}(v_0, r_0, \phi_0, \lambda_1)_0$ $\quad ecc(v_0) := TOF_{alg}(v_0, r_0, \phi_0, \lambda_1)_1$

Initial Conditions: $r_0 = 1.05 \cdot DU$ $\quad$ Alititude $:= r_0 - 1DU = 318.907 \cdot km$ $\quad \phi_0 = 0$

$hyperbolic := 1$ $\quad v_{inj} := 10.8kmps, 10.805kmps .. 11.2kmps$

Note: As the velocity increases above the minimum 10.8 kmps, the Time of Flight decreases and the trajectory shape changes from Elliptical to Hyperbolic.



Polar Plot of the Solution for the Patched Conic Lunar Approximation

$\nu := -90.002deg, -90.001deg..41deg$ $\chi := 39.5deg, 39.501deg..360deg$ $r(\nu) := \frac{a\cdot(1-e^2)}{1+e\cdot\cos(\nu+\gamma_0)}$

Note: λ_1 is not = 30 deg $\varphi := 33deg$ $Earth(\theta) := 1.5\sin(\theta+\varphi)$ $\theta := 0, 0.001..2\pi$

$r_m := 82$ $a_m := 1.5$ $r_{moon}(\theta,\varphi) := r_m\cdot\cos(\theta-\varphi) + \sqrt{a_m^2 - r_m^2\sin(\theta-\varphi)^2}$

Radius of Moon Sphere of Influence $r_{msi}(\theta,\varphi) := r_m\cdot\cos(\theta-\varphi) + \sqrt{10.4^2 - r_m^2\sin(\theta-\varphi)^2}$

$\xi := 0.05, 0.051..\varphi - 0.05$ $r_{m\_path}(\xi) := r_m$

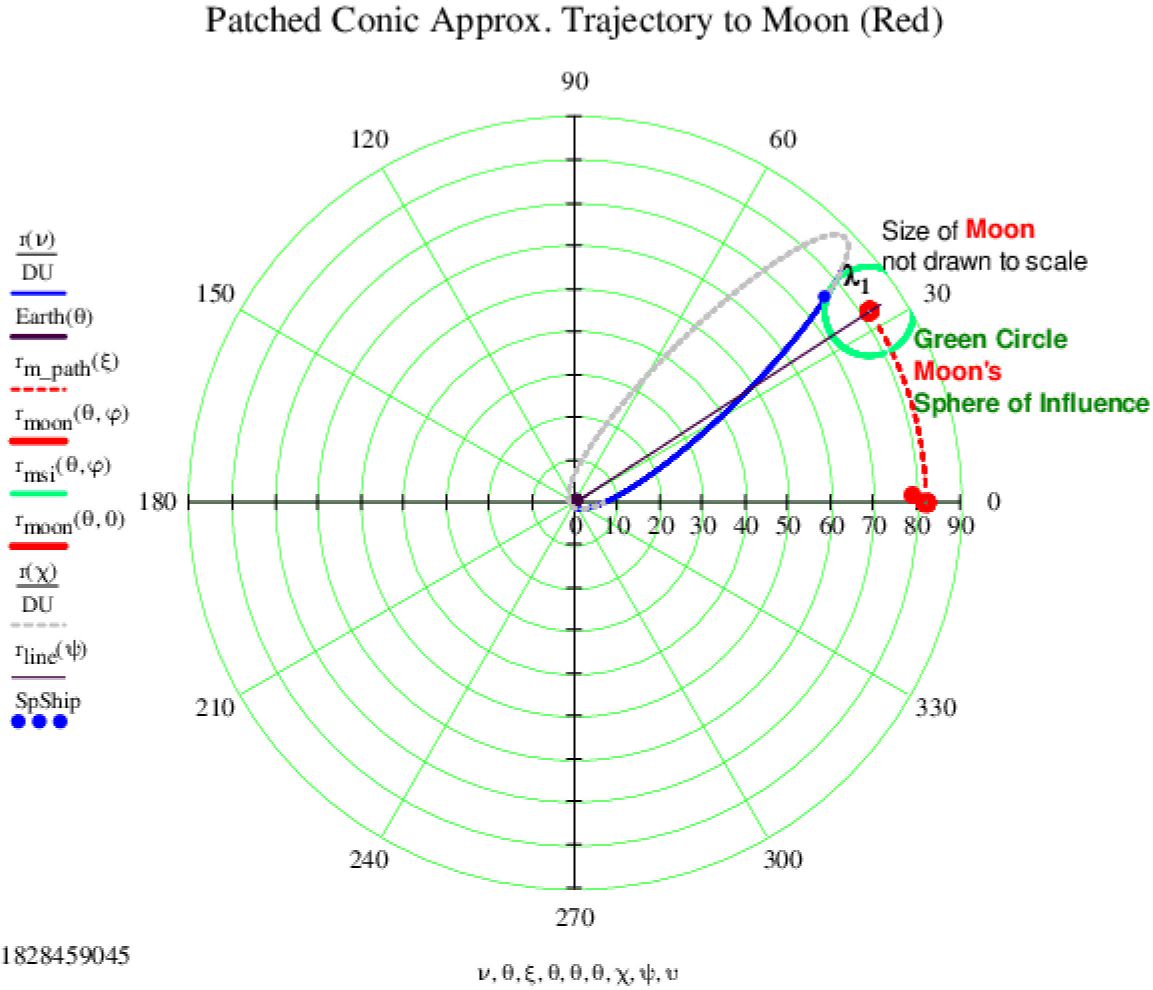
Point of Conic Patch
$SpShip := 75.5$

$\psi := 0, 0.0017365..\varphi$ $r_{line}(\theta) := \frac{0.1}{\sqrt{1-(1\cdot\cos(\theta-\varphi))^2}}$ $\upsilon := 39.5deg$

Polar Plot: Geocentric Frame - Earth at the Center

From the list of functions shown on the left of the plot below:

r(ν) shows the Trajectory Ellipse Conic Ptach in blue, Earth(θ) is at the center in black, $r_{moon}(\theta,\varphi)$ in red is the location of the moon at intercept $\varphi = 33^o$, $r_{msi}(\theta)$ is the circle in green of the moon's of sphere of influence, $r_{moon}(\theta,0)$ in red is the initial location of the moon at 0^o, $r_{m\_path}(\xi)$ is the dotted line path of moon from 0 to φ. r(χ) is the dotted line that shows the elliptical path back to the earth, and r_{line} is the red straight line from earth at center to the moon to show angle λ_1. SpCraft is where SpaceCraft enters the Moon's Sphere of Influence. Point of Conic Patch. Blue dot.



$e := 2.718281828459045$

IA. Apollo Free Return Trajectory: Simulation for CSM to Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

This 3 body gravitational solution for the FRT uses the Mathcad Differential Equation Solving Methodology discussed: arXiv:1504.07964
"Motion of the planets: the calculation and visualization in Mathcad", Valery Ochkov, Katarina Pisa

The aborted Apollo 13 mission was the only mission to actually turn around the Moon in a free-return trajectory.

Solve the Gravitational and Dynamics Equations for Earth, Moon, & CSM Trajecto

$kg := 1$ $m := 1$ $s := 1$ $N := 1$ $s := 1$ $min := 60s$ $hr := 3600s$ $kgf := 9.80665N$

$km := 1000m$ $kmps := km$ $kph := \frac{km}{hr}$ $mph := 0.447\cdot 10^{-3} kmps$ $G := 6.67384\cdot 10^{-11}\cdot\frac{N\cdot m^2}{kg}$

Run Simulation for 160 hrs **Apollo 11 Orbit 77 hrs**

$FRAME := 999$ $n_{ode} := 20000$ $n := 999$ $n_{plot} := 10000$ $t_{end} := \frac{160hr}{n+1}\cdot(FRAME+1)$ $t_{orb} = 81.44\ hr$

Time of Flight (TOF) = t_{orb}

Trajectory to Moon's Sphere of Influence

Initial x,y Velocity CSM **Radius of Earth** **Apogee to Moon**

$v_{0x} := 6.811kmps$ $v_{0y} := 6.356kmps$ $v_{CSM} := 9.317kmps$ $R_e := 6370km$ $d_{m\_ap} := 405500km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

Mass Start position Start Velocity

Earth, e; Moon, m; CSM, s

$$\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972\cdot 10^{24}kg & 0m & 0m & 0kph & 0kph \\ 7.347\cdot 10^{22}kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e+100km & R_e-100km & v_{0x} & v_{0y} \end{pmatrix}$$

Given **Solve Set of Differential Guidance Equations for 3 Body Problem of Earth, Moon, and CSM**

$x_e(0) = x_{e0}$ $x_e'(0) = vx_{e0}$ $y_e(0) = y_{e0}$ $y_e'(0) = vy_{e0}$

$$m_e\cdot x_e''(t) = \frac{G\cdot m_e\cdot m_m\cdot(x_m(t)-x_e(t))}{\left[\sqrt{(x_e(t)-x_m(t))^2+(y_e(t)-y_m(t))^2}\right]^3} + \frac{G\cdot m_e\cdot m_s\cdot(x_s(t)-x_e(t))}{\left[\sqrt{(x_e(t)-x_s(t))^2+(y_e(t)-y_s(t))^2}\right]^3}$$

$$m_e\cdot y_e''(t) = \frac{G\cdot m_e\cdot m_m\cdot(y_m(t)-y_e(t))}{\left[\sqrt{(x_e(t)-x_m(t))^2+(y_e(t)-y_m(t))^2}\right]^3} + \frac{G\cdot m_e\cdot m_s\cdot(y_s(t)-y_e(t))}{\left[\sqrt{(x_e(t)-x_s(t))^2+(y_e(t)-y_s(t))^2}\right]^3}$$

$x_m(0) = x_{m0}$ $x_m'(0) = vx_{m0}$ $y_m(0) = y_{m0}$ $y_m'(0) = vy_{m0}$

$$m_m\cdot x_m''(t) = \frac{G\cdot m_m\cdot m_e\cdot(x_e(t)-x_m(t))}{\left[\sqrt{(x_m(t)-x_e(t))^2+(y_m(t)-y_e(t))^2}\right]^3} + \frac{G\cdot m_m\cdot m_s\cdot(x_s(t)-x_m(t))}{\left[\sqrt{(x_m(t)-x_s(t))^2+(y_m(t)-y_s(t))^2}\right]^3}$$

$$m_m\cdot y_m''(t) = \frac{G\cdot m_m\cdot m_e\cdot(y_e(t)-y_m(t))}{\left[\sqrt{(x_m(t)-x_e(t))^2+(y_m(t)-y_e(t))^2}\right]^3} + \frac{G\, m_m\cdot m_s\cdot(y_s(t)-y_m(t))}{\left[\sqrt{(x_m(t)-x_s(t))^2+(y_m(t)-y_s(t))^2}\right]^3}$$

$x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$

$$m_s\cdot x_s''(t) = \frac{G\cdot m_s\cdot m_e\cdot(x_e(t)-x_s(t))}{\left[\sqrt{(x_s(t)-x_e(t))^2+(y_s(t)-y_e(t))^2}\right]^3} + \frac{G\cdot m_s\cdot m_m\cdot(x_m(t)-x_s(t))}{\left[\sqrt{(x_s(t)-x_m(t))^2+(y_s(t)-y_m(t))^2}\right]^3}$$

$$m_s\cdot y_s''(t) = \frac{G\cdot m_s\cdot m_e\cdot(y_e(t)-y_s(t))}{\left[\sqrt{(x_s(t)-x_e(t))^2+(y_s(t)-y_e(t))^2}\right]^3} + \frac{G\cdot m_s\cdot m_m\cdot(y_m(t)-y_s(t))}{\left[\sqrt{(x_s(t)-x_m(t))^2+(y_s(t)-y_m(t))^2}\right]^3}$$

IA. Free Return Trajectory: 3 Body Sim for CSM to the Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

Differential Equation Solver

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.316 \cdot \text{kmps}$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right] \qquad \frac{d_m(t_{orb})}{R_m} = 1.619$$

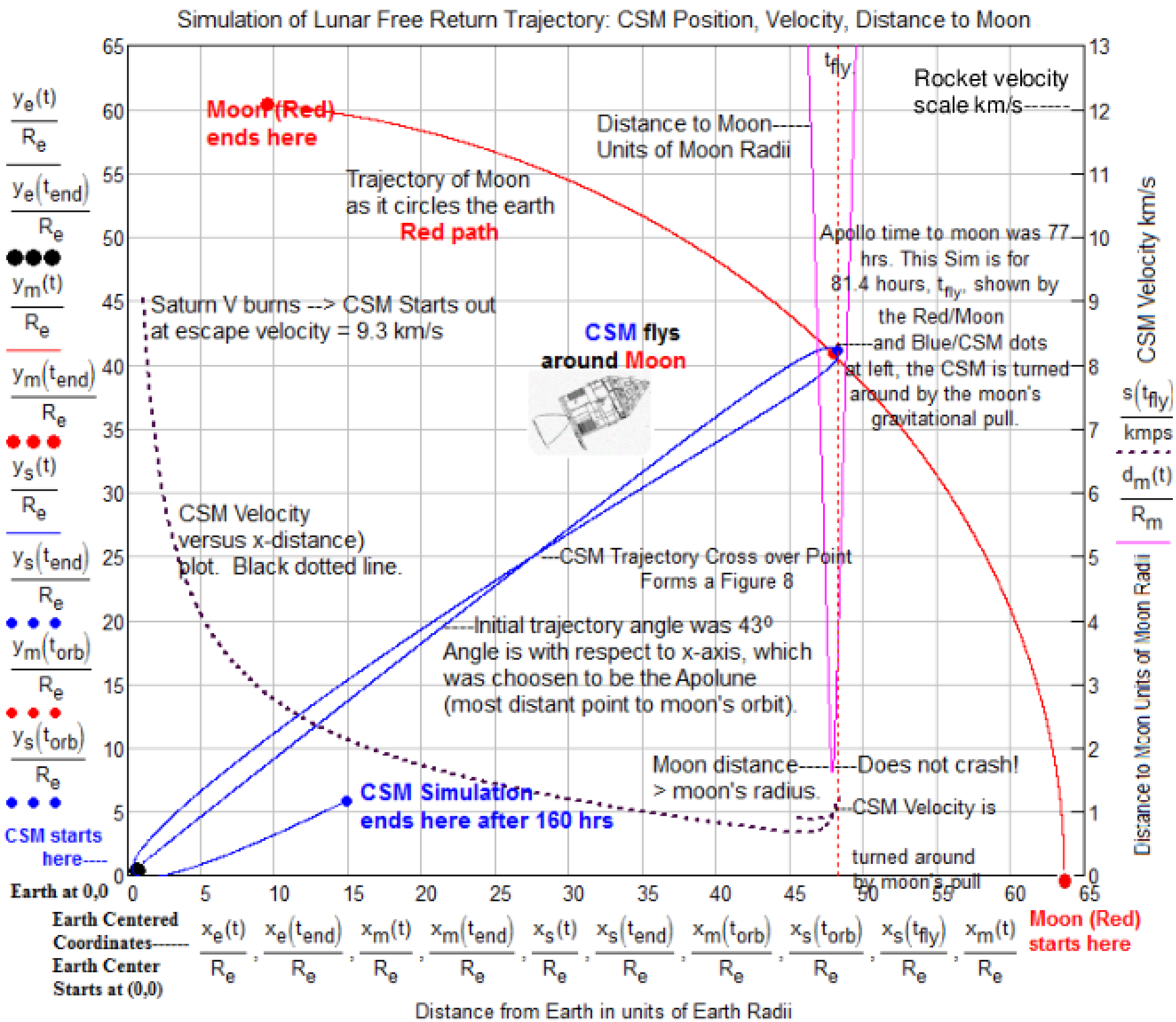
time t_{fly}, just beyond lunar fly by time $\quad$ **t_{flydot}**

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 90hr \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

Finding a Free Return Trajectory (FRT) is a little tricky. First, the trajectory must catch the moon at the exact place and time as travels around the earth and then after being swing around by the moon's gravity it must swing back and catch the earth in such a way as to go into earth orbit. This can present a problem for the Differential Equation Solver. This is a three body problem. A change in the CSM's trajectory is influenced by the pull the moon, which in turn is affected by the pull of the earth. The solver can easily fail to converge on a solution. A change in angle by 10 degrees can result in a large change in orbit time of 4.5 days. We also must check that CSM does not crash into moon.

Below is a plot of our FRT solution for the Apollo Trajectory. It shows the CSM's x,y position and velocity from earth to moon and back. .Note the figure 8 orbit of this Free Return. The Apollo 11 flight time to the moon was 77 hours. Our simulation is for 81.4 hours. Because of instabilities, convergence problems, etc. some trial and error was required.



IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon & Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

This Simulation Uses the Mathcad Differential Equation Solving Methodology discussed in: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

4-Body Reference Frame: Earth and moon are initially at 0,0 and the earth and sun are initially not moving.

$kg := 1$ $\quad m := 1$ $\quad s := 1$ $\quad N := 1$ $\quad s := 1$ $\quad min := 60s$ $\quad hr := 3600s$ $\quad kgf := 9.80665N$

$km := 1000m$ $\quad kmps := km$ $\quad kph := \frac{km}{hr}$ $\quad mph := 0.447 \cdot 10^{-3} kmps$ $\quad n_{plot} := 10000$

Run Simulation for 115 hrs $\quad$ **Apollo 11 Orbit 77 hr**

$FRAME := 999$ $\quad n_{ode} := 20000$ $\quad n := 999$ $\quad t_{end} := \frac{114.5hr}{n+1} \cdot (FRAME + 1)$ $\quad t_{orb} := 58.5hr$

Time of Flight (TOF) = t_{orb}

$G := 6.67384 \cdot 10^{-11} \frac{N \cdot m^2}{kg}$

Trajectory to Moon's Sphere of Influence $\quad$ Apolune

Initial x,y Velocity CSM $\quad$ **Radius of Earth** $\quad$ **Apogee to Moon**

$v_{0x} := 7.58kmps$ $\quad v_{0y} := 5.5kmps$ $\quad R_m := 1737.4km$ $\quad R_e := 6370km$ $\quad d_{m\_ap} := 405500km$

$t_{end} = 114.5\,hr$ $\quad v_{CSM} := \sqrt{v_{0x}^2 + v_{0y}^2}$ $\quad v_{CSM} = 9.365 \cdot kmps$ $\quad d_{e\_ap} := 152 \cdot 10^6 km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

e is Earth
a is Sun
m is Moon
s is CSM

$$\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_a & x_{a0} & y_{a0} & vx_{a0} & vy_{a0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kmps \\ 1.989 \cdot 10^{30} kg & -130 \cdot 10^6 km & -80 \cdot 10^6 km & 0kmps & 0kmps \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 110km & R_e - 96km & v_{0x} & v_{0y} \end{pmatrix}$$

Given $\quad$ **Set of Differential Guidance Equations for 4 Body Problem of Earth, Moon, and CSM**

$x_e(0) = x_{e0}$ $\quad x_e'(0) = vx_{e0}$ $\quad y_e(0) = y_{e0}$ $\quad y_e'(0) = vy_{e0}$ $\quad x_m(0) = x_{m0}$ $\quad x_m'(0) = vx_{m0}$ $\quad y_m(0) = y_{m0}$ $\quad y_m'(0) = vy_{m0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_S \cdot (x_S(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_S \cdot (y_S(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3}$$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_S \cdot (x_S(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G m_m \cdot m_S \cdot (y_S(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]}$$

$x_s(0) = x_{s0}$ $\quad x_s'(0) = vx_{s0}$ $\quad y_s(0) = y_{s0}$ $\quad y_s'(0) = vy_{s0}$ $\quad x_S(0) = x_{S0}$ $\quad x_S'(0) = vx_{S0}$ $\quad y_S(0) = y_{S0}$ $\quad y_S'(0) = vy_{S0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (x_S(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (y_S(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot x_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (x_e(t) - x_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (x_m(t) - x_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (x_s(t) - x_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot y_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (y_e(t) - y_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (y_m(t) - y_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (y_s(t) - y_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon and Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

Plot for Sim of 4-Body Free Return Traj: CSM to Moon and Back

Differential Equation Solver

Earth, Moon, Space Craft, Sun:

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_a \\ y_a \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_a \\ y_a \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.365 \cdot \text{kmps}$$

time t_{fly}, just beyond lunar fly by time **t flyby dot**

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 160hr \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

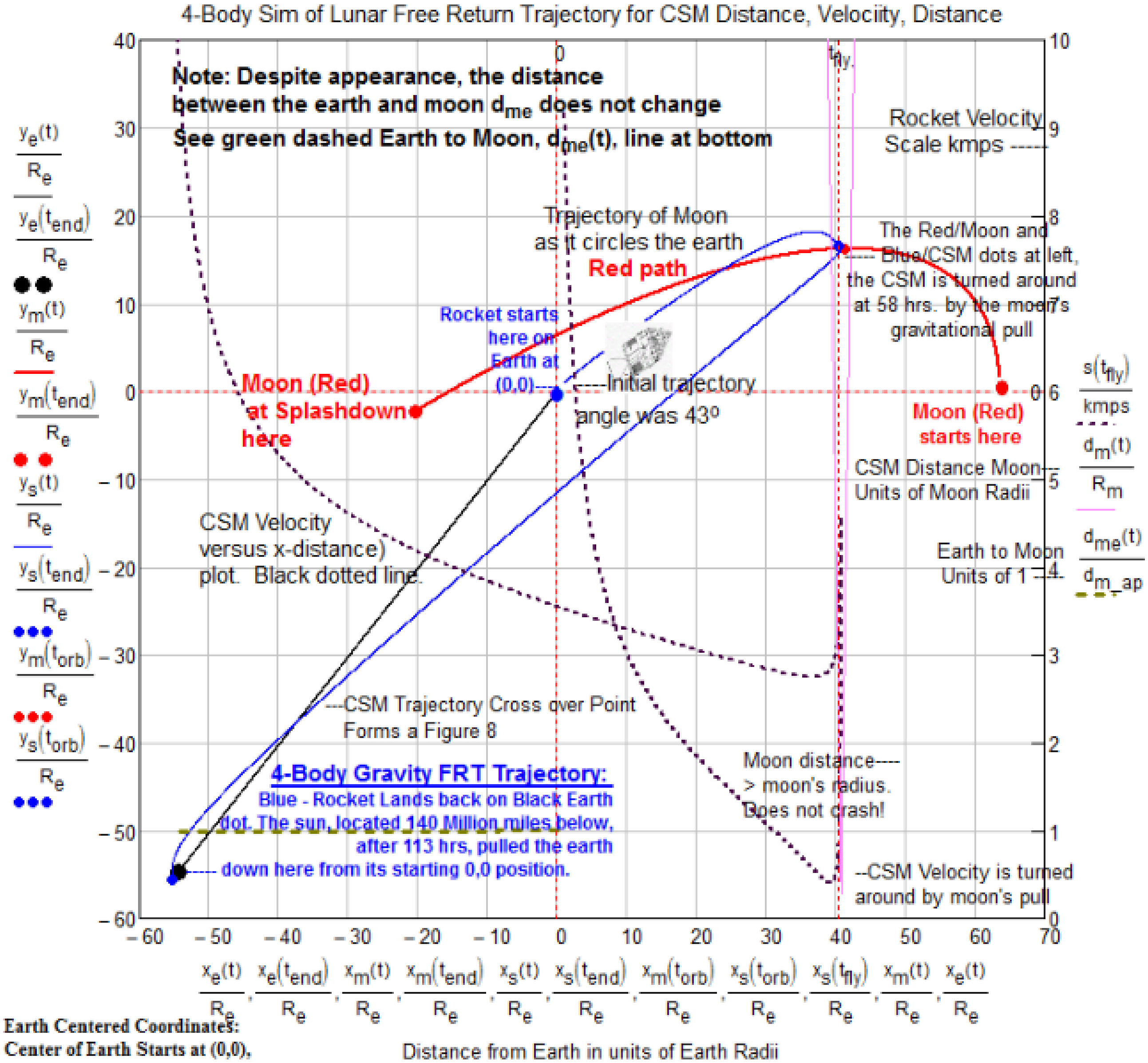
Distance from Earth to Moon

$$d_{me}(t) := \left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]$$

$$\frac{d_m(t_{orb})}{R_m} = 3.242$$



Earth Centered Coordinates: Center of Earth Starts at (0,0), but gravitational pull of sun, 94 million miles below-left of earth pulls the earth down & left from 0,0 so it ends at black dot above. The rocket (blue dot) lands back on earth 114 hours after launch.

Note: The radial velocity of the earth around the sun is 1° every 365 days or 1/365° per day. Our sim runs 114 hrs or 114/24 days. This results in (1/365°) x 114/24 or 7.5°. For the purpose of our illustration, we will ignore this added complexity. Think of this as a rotating reference frame, such as our experience of us living on a rotating earth

IC. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon and Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Sun at Center**

We examine 2 different versions of 4-Body Trajector & 2 Conditions of Moon Gravity: Turned Off and Turned (

In the first version, above, we put the earth at the center (0,0) and the sun is at 45° below (below left), 93 million miles aw
In the second below, we have the sun initially directly below the earth, 93 million miles away. The sun is at the center (0,(
The earth revolves around the sun with a velocity of 30 km/s and the moon revolves around the earth at 0.94 km/s. The gravitational pull of the sun pulls both the earth and moon downwards toward the sun. We solve the system of differentia equations for the spacecraft, earth, moon, and Sun. Then we do a change of variable to make the earth at the center. **Putting the sun at the center gives a more accurate simulation**. We have a far closer match to Apollo mission times. Apollo Times to Moon and Back to Earth: 77 and 142 hrs. Simulation Times to Moon and Earth: 77 and 138 hrs.

Run Simulation for 180 hrs **Apollo 11 Orbit 77 hrs**

$FRAME := 999$ $n_{ode} := 20000$ $n := 999$ $n_{plot} := 10000$ $t_{end} := \frac{138hr}{n+1} \cdot (FRAME + 1)$ $t_{orb} := 77hr$

Time of Flight (TOF) $= t_{orb}$

$G := 6.67384 \cdot 10^{-11} \frac{N \cdot m^2}{kg}$

Trajectory to Moon's Sphere of Influence

Initial x,y Velocity CSM **Radius of Earth** **Apogee, Perogee to Moon**

$v_{0x} := -0.0kmps$ $v_{0y} := 9.234kmps$ $R_m := 1737.4km$ $R_e := 6370km$ $d_{m\_ap} := 405500km$

$v_{CSM} := \sqrt{v_{0x}^2 + v_{0y}^2}$ δv_{CSM} 9.229 kmps $d_{e\_ap} := 147.1 \cdot 10^6 km$ $d_{m\_pe} := 3.45 \cdot 10^8 m$

Define Gravitational and Dynamics Equations for Earth, Moon, Sun and Spacecraft (CSM)

e is Earth
S is Sun
m is Moon
s is CSM

$$\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_S & x_{S0} & y_{S0} & vx_{S0} & vy_{S0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & d_{e\_ap} & -29 kmps & 0 kmps \\ 1.989 \cdot 10^{30} kg & 0m & 0m & 0kmps & 0kmps \\ 7.347 \cdot 10^{22} kg & 0km & d_{e\_ap} + d_{m\_ap} & -29.983kmps & 0kmps \\ 13600kg & R_e + 334.4km & d_{e\_ap} + R_e - 90km & v_{0x} - 29kmps & v_{0y} \end{pmatrix}$$

Given **Set of Differential Guidance Equations for 4 Body Problem of Earth, Moon, Sun, and CSM**

$x_e(0) = x_{e0}$ $x_e'(0) = vx_{e0}$ $y_e(0) = y_{e0}$ $y_e'(0) = vy_{e0}$ $x_m(0) = x_{m0}$ $x_m'(0) = vx_{m0}$ $y_m(0) = y_{m0}$ $y_m'(0) = vy_{m0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_S \cdot (x_S(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_S \cdot (y_S(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3}$$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_S \cdot (x_S(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G m_m \cdot m_S \cdot (y_S(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3}$$

$x_s(0) = x_{s0}$ $x_s'(0) = vx_{s0}$ $y_s(0) = y_{s0}$ $y_s'(0) = vy_{s0}$ $x_S(0) = x_{S0}$ $x_S'(0) = vx_{S0}$ $y_S(0) = y_{S0}$ $y_S'(0) = vy_{S0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (x_S(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (y_S(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot x_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (x_e(t) - x_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (x_m(t) - x_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (x_s(t) - x_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot y_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (y_e(t) - y_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (y_m(t) - y_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (y_s(t) - y_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

IC. 4-Body Sim of Free Return Trajectory: CSM to Moon & Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w Sun **at Center**

Best Match to Apollo 11: Time to Moon Apollo - 77 hrs, Sim -77 hrs. To Earth Apollo -142 hrs Sim - 138 hrs

Differential Equation Solver

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_S \\ y_S \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_S \\ y_S \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}{}^2 + v_{0y}{}^2} = 9.234 \cdot \text{kmps}$$

time t_{fly}, just beyond lunar fly by time

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. t_{end}$$

$$vx_s(t) := \frac{d}{dt} x_s(t) \qquad vy_s(t) := \frac{d}{dt} y_s(t) \qquad s(t) := \sqrt{\left(vx_s(t) + 29\text{kmps}\right)^2 + vy_s(t)^2}$$

$$s(t_{orb}) = 3.005 \cdot \text{kmps}$$

Distance to the Center of the Moon

$$d_{ms}(t) := \left[\sqrt{\left(x_m(t) - x_s(t)\right)^2 + \left(y_m(t) - y_s(t)\right)^2}\right] \qquad \frac{d_{ms}(t_{orb})}{R_m} = 3.616 \qquad d_s := d_{e\_ap}$$

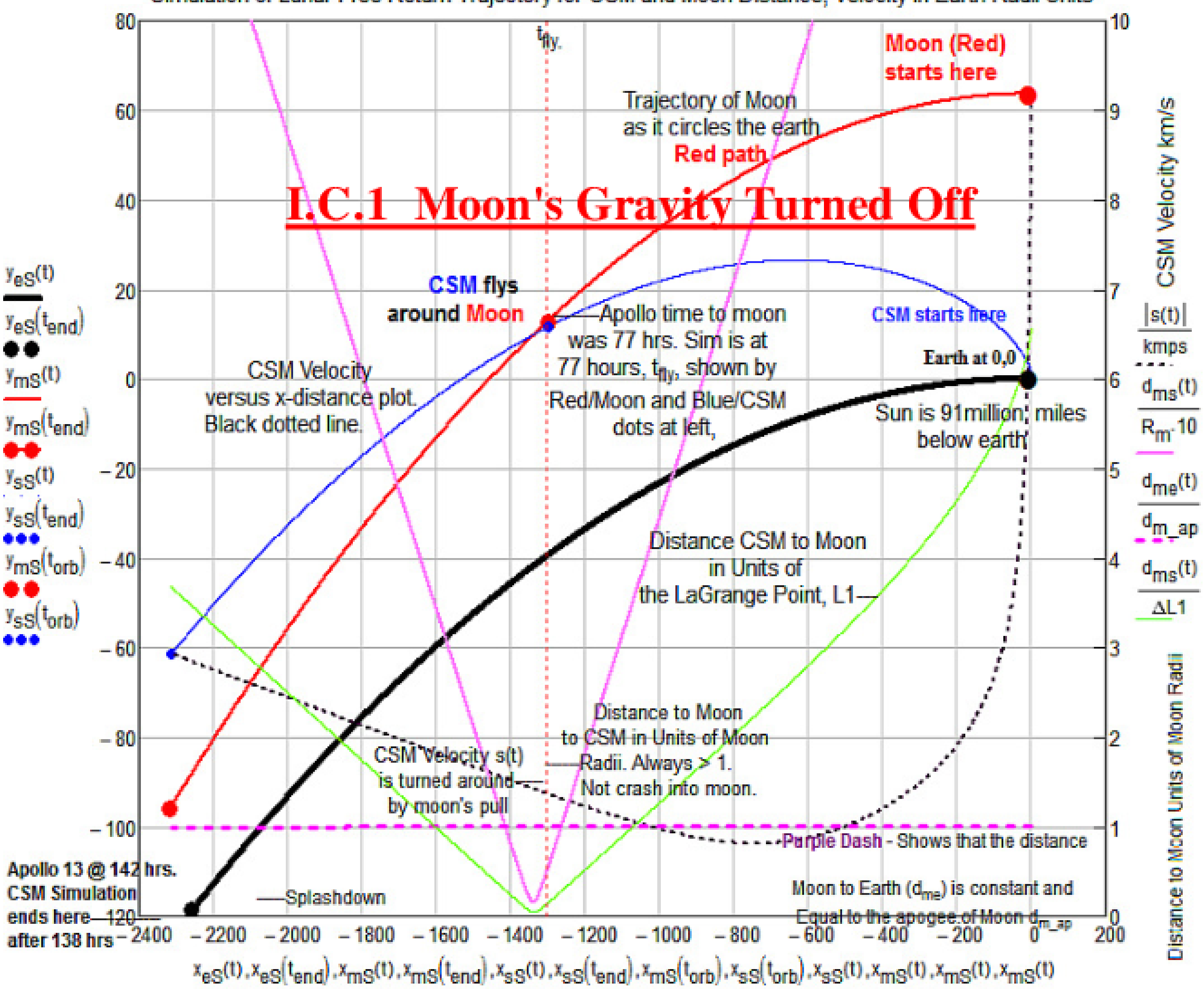
Change of Coordinates: Change to Frame Where the Earth is at Center & Use Units of Earth Radius

$$x_{eS}(t) := \frac{x_e(t)}{R_e} \qquad y_{eS}(t) := \frac{y_e(t) - d_s}{R_e} \qquad x_{mS}(t) := \frac{x_m(t)}{R_e} \qquad y_{mS}(t) := \frac{y_m(t) - d_s}{R_e} \qquad x_{sS}(t) := \frac{x_s(t)}{R_e}$$

t flyby dot

$$d_{me}(t) := \left[\sqrt{\left(x_m(t) - x_e(t)\right)^2 + \left(y_m(t) - y_e(t)\right)^2}\right] \qquad t_{fly.} := x_{sS}(t_{orb}) \qquad t_{fly.} = -1.3 \times 10^3 \qquad y_{sS}(t) := \frac{y_s(t) - d_s}{R_e}$$

Simulation of Lunar Free Return Trajectory for CSM and Moon Distance, Velocity in Earth Radii Units



Distance of Earth, Moon, and Apollo from Earth in units of Earth Radii

I.C.2 Moon's Gravity Turned On: Free Return Trajectory

Under the influence of the moon's gravity, the CSM now loops around the moon and is redirected back to the earth on a FRT.

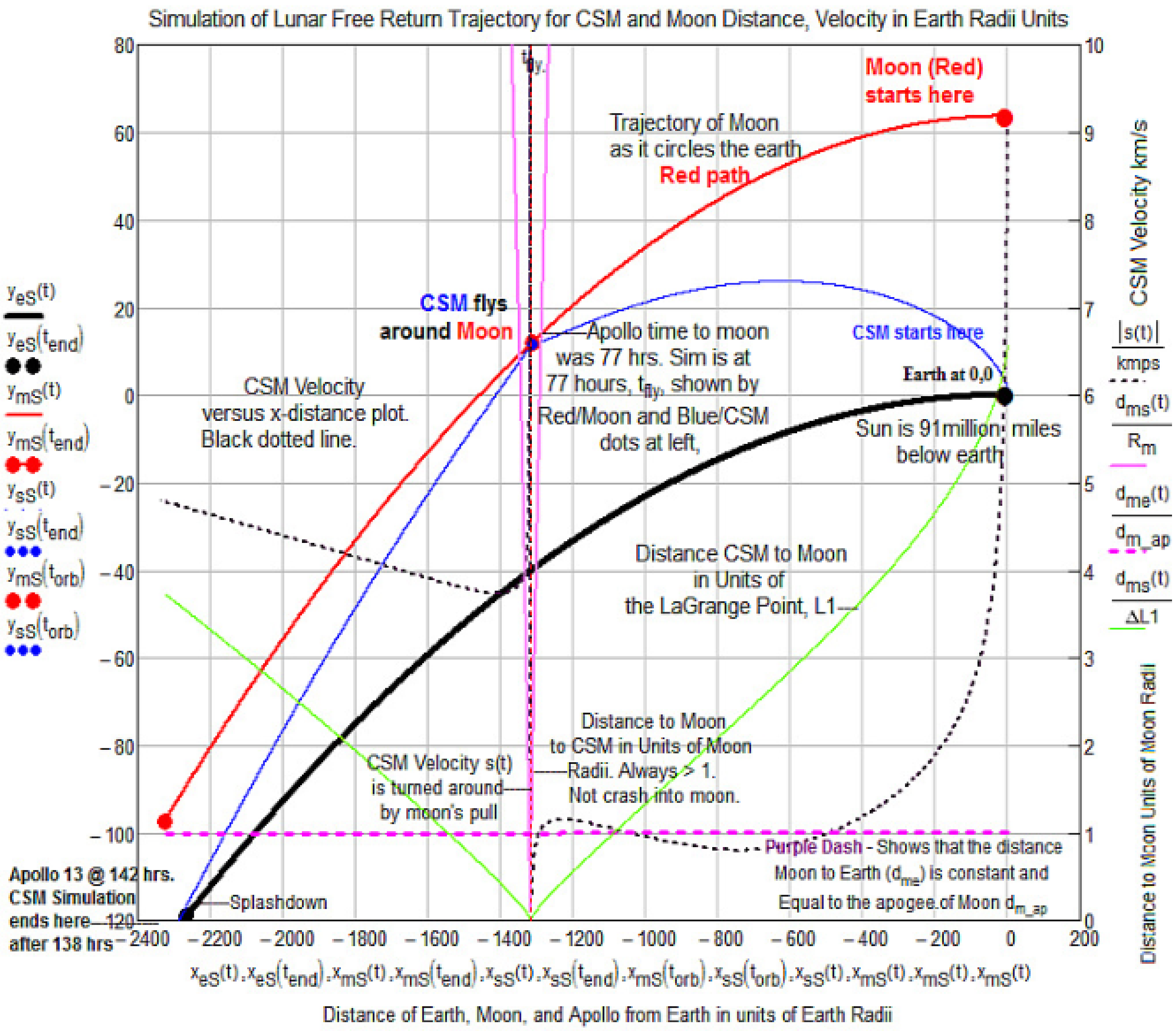
Note that the scale is very distorted. The x direction is 100X larger than the y direction. This is a result of the fact that the sun is located directly below and the earth and therefore the CSM are moving at 30 km/s or 67,000 mph around the sun toward the left or -x direction.

Legend - Distance (Units of Earth Radius) versus x distance

Black: Earth y Distance, $y_{eS}(t)$
Red: Moon y Distance, $y_{mS}(t)$
Blue: CSM y Distance, $y_{sS}(t)$
Solid Purple: CSM to Moon Distance, $d_{ms}(t)$, Units of Moon Radius
Dotted Purple: Earth to Moon Distance, $d_{me}(t)$, Units of Distance of Earth to Moon - shows that orbit distance does not change
Dotted Green: Moon to CSM, $d_{ms}(t)$ Units of Distance to L1 Point from the Moon

Legend - Velocity

Dotted Black: CSM Speed, s(t)



II. Table of Basic Saturn IV Engine Parameters

Stages	m_{fuel}	Frame	Burn Duration		Exhaust Velocity	Burn Rate	Thrust
	($x10^6$kg)	($X10^6$kg)	(s)		(m/s)	(kg/s)	($x10^6$N)
1st stage- 5 F1s	2.04	0.136	165	Kerosene/LOX	2456	13600	33.4
2nd stage- 2 J2s	0.428	0.0432	360	LH2/LOX	4220	1190	5.02
S-IVB Orbit-1 J2	0.0356		165	LH2/LOX	4630	216	1
S-IVB TransLunar	0.0674	0.0174	312	LH2/LOX	4630	216	1
Park Orbit Payload		0.118					
Lunar Module	8200kg	2134kg		Aerozine/N2O4	3005	I_{sp}=311s	45kN
Total	**2.571**	**0.1966**					

Apollo 11 Day 1: First Stage Liftoff- Lower Atmosphere transport to 38 miles up.
The Saturn V has three states. The first Stage, which is the largest, consisted of five Rocket dyne F-1 engines, producing 33.4x 10^6 N (7.5 million pounds) of thrust. The duration of the burn is 165 seconds. The fuel was RP-1 refined kerosene, and the oxidizer was liquid oxygen (LOX). The center fifth engine was turned off early to limit the acceleration to 4 g's on the astronauts. NASA reported that the **final velocity of Stage I was of 2.76 km/s**. The spacecraft does not maintain a constant vertical launch pitch angle trajectory, but angles downward into a near horizontal orbital trajectory. The initial vertical launch had a 10 s delay, then the pitch angle decreases linearly with time. This pitch profile vs, time is simulated. See graph below. This affects vertical acceleration, avert. At the end of the first Stage the spacecraft is downrange about 58 miles (93 km) with an altitude of 38 miles. Approximate travel distance of about 69 miles.

Specific Impulse, Isp, is defined as the number of pound-s of impulse (thrust times duration) given by 1 kg mass of propellant. Kerosene (RP-1) generates an Isp in the range of 270 to 360 seconds, while liquid hydrogen (LH2) engines achieve 370 to 465 seconds.

Apollo 11 Statistics - Ground Ignition Mass: $6477875\text{lb} = 2.938\times 10^6\,\text{kg}$

Saturn V S-I Engine Parameters:

$m_{fuel} := 2.04\cdot 10^6\text{kg}$ $\quad m_{tot} := 2.77\cdot 10^6\text{kg}$ $\quad m_{frame} := 1.36\cdot 10^5\text{kg}$ $\quad R := 1.289\cdot 10^4\cdot\frac{\text{kg}}{\text{s}}$

Engine Thrust Change in Momentum, T: $T = d(mv)/dt = v_{exhaust} \times dm/dt = v_{exhaust} \times$ Fuel Burn Rate

$v_{exhaust} := 2715\frac{\text{m}}{\text{s}}$ $\quad v_{ex} := v_{exhaust}$ $\quad v_{ex} = 8.907\times 10^3\cdot\frac{\text{ft}}{\text{s}}$ $\quad T := v_{ex}\cdot R = 3.5\times 10^7\,\text{N}$

Approximate the rate of fuel consumption, mass(t), with a linear model: $\quad \text{mass1}(t) := m_{tot} - R\cdot t$

III. Create a Model for Pitch Angle, θ

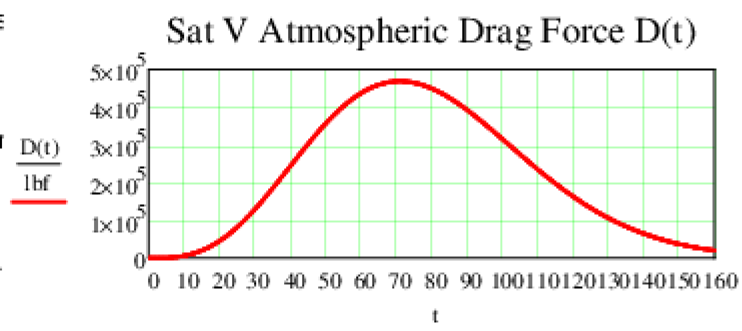
of the rocket's trajectory's during Stage 1 Burn. This angle then determines the resulting g component of the earth's gravitational pull, on the rocket, **in the direction of thrust. Call this g_{thrust}.** Define a parameter, τ, that can be used to adjust the rate and final angle during burn. Adjust τ to match final velocity equal to NASA's Stage 1 velocity. Let the pitch angle, θ, decrease linearly with burn time, from initially 90 degrees vertical to ~ 30 degrees horizontal at the end of Stage 1 burn. D(t) approximates the Drag. The resultant horizontal acceleration in the direction of travel is then: a(t) - g_{thrust}

$\tau := 400\text{s}$ $\quad 90^o = \pi/2$ radians. $\quad \theta(t) := \frac{\pi}{2}\cdot\left[1 - \Phi(t - 10\text{s})\cdot\left(0.31 + \frac{t}{\tau}\right)\right]$ $\quad S(x) := 9.027x\cdot\log\left(1 + x^2\right)\exp\left(-x^2\right)$

IV. Saturn V Atmospheric Drag Data

$D(t) := 4.6\cdot 10^5\,S\left(\frac{t}{65}\right)\cdot\text{lbf}$

The AS-503 Flight Evaluation Report gives a graph of Apollo 8 Dynamic Pressure versus **flight time**. It is approximately equal to 460,000 pounds at 70s, drops off close to zero at 160s, and has the shape shown at right." The atmosphere ends at about 100km or about 200s into the flight. Thus, we approximate the **Saturn V Atmospheric Drag Force for Apollo Missions, D(t),** as function of recorded time as shown at right.



V. Stage I Simulation Equations for Pitch, Acceleration, Velocity, and Distance

$t_{burn1} := 165s$

$g_{thrust1}(t) := g \cdot \sin(\theta(t))$

NASA's velocity was 2760 m/s

Sim Stage 1:

$a1(t) := \dfrac{T - D\left(t \cdot s^{-1}\right)}{mass1(t)}$

$v1_{thrust}(t) := \int_0^t \left(a1(t) - g_{thrust1}(t)\right) dt$

$v1_{thrust}(165s) = 2.763 \times 10^3 \dfrac{m}{s}$

$a1(0) = 1.288 \cdot g$ $a1(11s) = 1.356 \cdot g$ $a1\left(t_{burn1}\right) = 5.538 \cdot g$ $v1_{thrust}(69s) = 1.03 \times 10^3 \cdot mph$

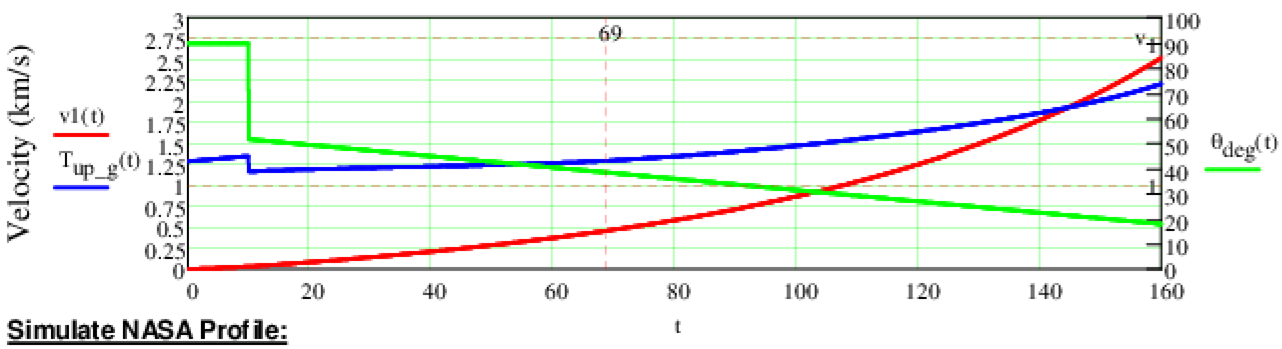
$v1_{final} := v1_{thrust}\left(t_{burn1}\right)$

VI. Graph Velocity, Vertical thrust, Tup. Check that Tup > 1 g as Pitch θ is Reduced

This is required to make certain that the vertical velocity does not decrease due to the pull of gravity

$T_{up\_g}(t) := a1(t) \cdot \sin(\theta(t)) \cdot g^{-1}$ $T_{up\_g}(10.1s) = 1.168$

Velocity (red), Upward Thrust (blue) and Pitch (green) vs. time



Simulate NASA Profile:

10 s delay to Pitchover Maneuver Lange ~ pitch from 45º to 20º to horizontal. Note: The above Thrust Angle Profile was created by approximating a graph of NASA 's pitch angle profile for State 1 launch.

VII. Calculate Stage 1 Altitude

NASA's Total Distance from launch point by Stage 1: $Straight\_Line := \sqrt{(58mile)^2 + (38mile)^2} = 69.34 \cdot mile$

$38mile = 61.155 \cdot km$

The calculated thrust distance is 89 miles, approximately equal to NASA's Stage 1 Travel Distance of 69 miles.

$Curved\_Path := \int_0^{t_{burn1}} \int_0^t a1(t) - g_{thrust1}(t)\, dt\, dt$

$Curved\_Path = 87.087 \cdot mile$

NASA Stage 1 altitude **38 miles**.

Calculate the Stage 1 Altitude

$alititude := \int_0^{t_{burn1}} v1_{thrust}(t) \cdot \sin(\theta(t))\, dt$ $alititude = 48.874 \cdot mile$

Calculate the Stage 1 Specific Impulse, I_{SP}:

Total Impulse, I $I := T \cdot t_{burn1} = 5.774 \times 10^9 \dfrac{m \cdot kg}{s}$ Specific Impulse, Isp $I_{sp} := \dfrac{I}{m_{fuel} \cdot g}$ $I_{sp} = 288.64s$

Tsiokolvsky's Equation for Stage 1 Δv: $\Delta v := v_{exhaust} \cdot \ln\left(\dfrac{m_{tot}}{m_{tot} - m_{fuel}}\right)$ $\Delta v = 3.621 \times 10^3 \dfrac{m}{s}$

VIII. Stage IIB Burn: Velocity Calculation - Assuming 20 Deg Ascent Angle & ~ 0.35g

NASA's velocity at the end of Stage 2 was **6995 km/s** at 191 km altitude. At this altitude, drag is not significant.

$m_{fuel} := 4.28 \cdot 10^5 kg$ $m_{tot2} := 5.94 \cdot 10^5 kg$ $v_{ex2} := 4220 \frac{m}{s}$ $R := 1190 \cdot \frac{kg}{s}$ $t_{burn2} := 360s$

$\theta_{deg}(1) = 90$ $g \cdot \sin\left(\frac{32}{90} \cdot \frac{\pi}{2}\right) = 0.53 \cdot g$

$g_{thrust1}(199s) = 0.298 \cdot g$ $\left(1 - \frac{199s}{\tau}\right) \cdot 90 = 45.225$ $v_{ex2} = 1.385 \times 10^4 \cdot \frac{ft}{s}$ $T2 := v_{ex2} \cdot R = 5.022 \times 10^6 N$

$mass2(t) := m_{tot2} - R \cdot t$ $a2(t) := \frac{T2}{mass2(t)} - g_{thrust1}(191s)$ $v2(t) := v1_{final} + \int_0^t a2(t)\, dt$

$v2_{final} := v2(t_{burn2})$

$g_{thrust1}(191s) = 0.328 \cdot g$ $a2(0s) = 0.534 \cdot g$ $a2(t_{burn2}) = 2.765 \cdot g$ **Sim Stage IIB:** $v2_{final} = 6.997 \times 10^3 \frac{m}{s}$

IX. Calculate the Required Velocity to go into Parking Orbit at Altitude of 191 km

$G := 6.67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$ $M_e := 5.972 \cdot 10^{24} kg$ $R_e := 6370km$ $v_{orbital} := \sqrt{G \cdot \frac{M_e}{R_e + 191km}}$

$v_{orbital} = 7.792 \cdot \frac{km}{s}$

Earth's Escape Velocity at 50,000 km C- C Distance:

$v_{50Mm} := \sqrt{\frac{2G \cdot M_e}{50000km}} = 3.992 \cdot \frac{km}{s}$

Earth's Escape Velocity from Surface of Earth:

$v_{escape} := \sqrt{\frac{2G \cdot M_e}{R_e}} = 11.183 \cdot \frac{km}{s}$

X. Day 1: Stage IVB Burn #1 - Parking Orbit: Velocity Calculation

Stage IVB is close to the altitude and orbital velocity, where gravity, rather than pulling the spacecraft down, is providing the centripetal acceleration needed to keep the spacecraft in a curved or orbital path around the earth. This is the Apollo 11 earth parking orbit.
It will make two trips in orbit around the earth before embarking towards the moon.

At an altitude of 191.2 km, Apollo 11 went into a parking orbit.
The stated NASA stage 3 velocity was **7.791 km/s.**

$m_{fuel} := 3.56 \cdot 10^4 kg$ $m_{tot3A} := 1.23 \cdot 10^5 kg$ $v_{ex3} := 4630 \frac{m}{s}$ $R := 216 \cdot \frac{kg}{s}$ $t_{burn3A} := 150s$

$mass3A(t) := m_{tot3A} - R \cdot t$ $v_{ex3} = 1.519 \times 10^4 \cdot \frac{ft}{s}$ $T := v_{ex3} \cdot R = 1 \times 10^6 N$

$mass3A(t_{burn3A}) = 9.06 \times 10^4 kg$ $a3A(t) := \frac{T}{mass3A(t)} - g_{thrust1}(165s)$ $v3A(t) := v2_{final} + \int_0^t a3A(t)\, dt$

$v3A_{final} := v3A(t_{burn3A})$

$a3A(0s) = 0.407 \cdot g$ $a3A(t_{burn3A}) = 0.703 \cdot g$

NASA stage 3 velocity was: **7.791 km/s.**

$v3A_{final} = 1.743 \times 10^4 \cdot mph$ **Sim Stage 3A Parking Orbit:** $v3A_{final} = 7.791 \cdot \frac{km}{s}$

XI. Orbital Mechanics - Estimate Velocity Required for Trans-Lunar Injection, TLI

See for example: ttp://www.braeunig.us/space/orbmech.htm#position

Stage IVB Burn: Fall to the Moon or Trans-Lunar Injection, TLI.

For the Apollo lunar missions, the re-startable J-2 engine in the third (S-IVB) stage of the Saturn V rocket was used to perform TLI. Apollo data states that the TLI burn provided 3.05 to 3.25 km/s (10,000 to 10,600 ft/s) of Δv, at which point the spacecraft was traveling at approximately **10.8 km/s** (34150 ft/s) relative to the Earth.

The final burn causes the orbit to change from circular (constant radius) to one that is elliptical. The final increase in velocity starts the TLI as the vehicle moves from the circular path around the earth to the orbit of the furthest point or the largest radius of the TLI. The Apollo 11 trip to the moon took 51 hours and 49 minutes. The average distance to the moon is 238,855 miles (384,400 km). To achieve orbital transfer to the moon, the vehicle must move within the **Lagrangian Point**, where the gravitational pull of the earth is equal to that of the moon. This distance from the earth is 326,054 km.

Refer to the Figure of the trajectory on page 6.

Flight Data for Apollo 11 TLI: Earth Orbit Insertion Space Fixed Velocity $25568 \frac{ft}{s} = 7.793 \cdot \frac{km}{s}$

Sim: Apollo Data TLI Space Fixed Velocity: $35545 \frac{ft}{s} = 10.834 \cdot \frac{km}{s}$

Estimate the Minimum Velocity Required for Orbit around the Moon

The minimum can be estimated by determining the change in Gravitational Potential Energy from a parking orbit 100 km above the earth to Lagrange Point 1 (LP1), located between the earth and moon, where the pull of the earth equals the pull of the moon. Lagrange Points are positions in space where the gravitational forces of a two body system like the Sun and the Earth produce enhanced regions of attraction and repulsion. There are 5 Earth-Moon Lagrange Points.

Distances

Earth to Moon — Earth to Lagrange Pt 1

$M_{earth} := 5.97 \cdot 10^{24} kg$ $\quad M_{moon} := 7.348 \cdot 10^{22} kg$ $\quad D_{EtoM} := 34400km$ $\quad d_{LP} := 326400km$

$m_{TLI} := 5.616 \times 10^{6}\ kg$

Radii of Earth and Moon

$R_{earth} := 6371km$ $\quad R_{moon} := 1079mile$

First, Calculate Change in Gravitational Potential Energy to move from Parking Orbit to Lagrange Point 1, ELP1

$$V_{earth} := -G \cdot m_{TLI} \cdot \left[\frac{M_{earth}}{R_{earth}} + \frac{M_{moon}}{D_{EtoM} - (R_{earth} + 100km)} \right] \qquad V_{LP1} := -G \cdot m_{TLI} \cdot \left(\frac{M_{earth}}{d_{LP}} + \frac{M_{moon}}{D_{EtoM} - d_{LP}} \right)$$

Gravitational PE from Earth Orbit to Lagrange Point 1, E_{LP1} $\quad E_{LP1} := V_{LP1} - V_{earth} = 3.452 \times 10^{14} J$

From this Change in Gravitational Energy, ETP, we can get required velocity from Parking Orbit, to moon, v_{PO_Mn}.

From Conservation of Energy,
Needed Kinetic Energy = E_{LP1}
This gives Required velocity, v_{TLI}

$$\frac{1}{2} m_{TLI} \cdot v_{PO\_Mn} = E_{LP1} \qquad v_{PO\_Mn} := \sqrt{\frac{2 \cdot E_{LP1}}{m_{TLI}}} \qquad v_{PO\_Mn} = 11.088 \cdot \frac{km}{s}$$

XII. Day 1: Stage IVB Burn #2 - Trans-Lunar Injection, TLI. Earth Orbit to Moon

Adjust g for pitch and distance from the earth.

$$g_{thrust3B} := g \cdot \frac{R_e^{\,2}}{(R_e + 100km)^2} \sin\left(\frac{\pi}{2} \cdot \frac{1}{90}\right)$$

$Payload := 7.1 \cdot 10^{4} kg$ $\quad T = 1 \times 10^{6} N$

$m_{fuel} := 6.74 \cdot 10^{4} kg$ $\quad m_{tot3B} := m_{fuel} + Payload$ $\quad R := 216 \cdot \frac{kg}{s}$ $\quad t_{burn3B} := 312s$

$$mass3B(t) := m_{tot3B} - R \cdot t \qquad a3B(t) := \frac{T}{mass3B(t)} - g_{thrust3B} \qquad v3B(t) := v3A_{final} + \int_{0s}^{t} a3B(t)\, dt$$

$a3B(0s) = 0.72 \cdot g$ $\quad a3B(t_{burn3B}) = 1.419 \cdot g$

$mass3B(t_{burn3B}) = 7.101 \times 10^{4}\ kg$

Apollo TLI Data: 10.8 km/s

Sim Stage 3B TLI: $v3B_{final} = 10.829 \cdot \frac{km}{s}$

XIII. Graph the Velocity and Acceleration (gs) Profile of the Flight

$t_{burn1} = 165\,s$ $\quad t_{burn2} := 360s$ $\quad t_{burn3A} = 150\,s$ $\quad t_{burn3B} := 312$

$kmps := \frac{km}{s}$ $\quad v2(t) := v2(t \cdot s) \cdot kmps^{-1}$ $\quad v3A(t) := v3A(t \cdot s) \cdot kmps^{-1}$ $\quad v3B(t) := v3B(t \cdot s) \cdot kmps^{-1}$

$a1(t) := \frac{a1(t \cdot s)}{g}$ $\quad a2(t) := \frac{a2(t \cdot s)}{g}$ $\quad a3A(t) := \frac{a3A(t \cdot s)}{g}$ $\quad a3B(t) := if\left(t < 312, \frac{a3B(t \cdot s)}{g}, 0\right)$

$m_1(t) := mass1(t \cdot s) \cdot Mkg$ $\quad m_2(t) := mass2(t \cdot s) \cdot Mkg$ $\quad m_3(t) := mass3A(t \cdot s) \cdot Mkg$ $\quad m_4(t) := mass3B(t \cdot s) \cdot Mkg$

$$v_{SV}(t) := if(t < 165, v1(t), if(t < 525, v2(t - 165), if(t < 675, v3A(t - 525), if(t < 987, v3B(t - 675), 10.83))))$$

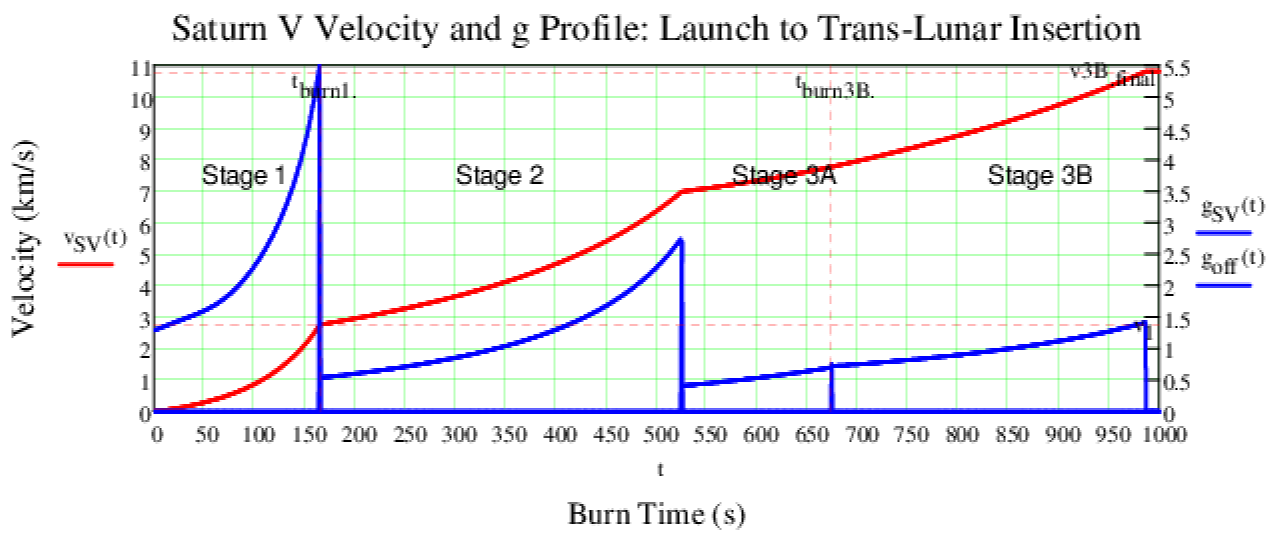
$$g_{SV}(t) := if(t < 165, a1(t), if(t < 525, a2(t - 165), if(t < 675, a3A(t - 525), a3B(t - 675))))$$

$$m_{SV}(t) := if(t < 165, m_1(t), if(t < 525, m_2(t - 165), if(t < 675, m_3(t - 525), m_4(t - 675))))$$

$t_{burn1.} := 165$ $\quad t_{burn3B.} := 675$ $\quad v3B_{.final} := v3B_{final} \cdot kmps^{-1}$

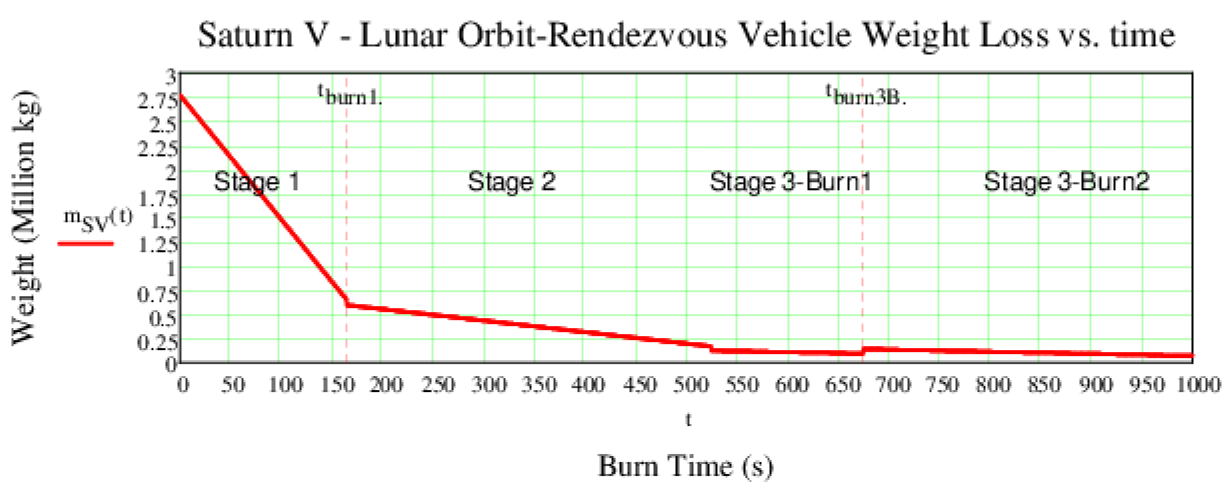
Note: $g_{off}(t)$ creates blue vertical markers at 165, 525, 675, and 987 s at completion of stage burns.

Simulation of the Four Saturn V Burns: Acceleration and Velocity



Note: The above acceleration is the net acceleration of the vehicle and not the net g thrust.
The g thrust of the engine is approximately 1 g greater than the net g force that moves the vehicle.

Sim: Weight Loss of 4 Saturn V Burns - Fuel Burns & Stage Jettisons



The above illustrates the LOR Strategy, where a large part of the weight is jettisoned during the flight.

XIV. Day 2-4: Command/Service Module Engine for Lunar & Earth Orbits - Lunar Module

Service Module	Mass	Mass	t_{final}	Fuel	$v_{exhaust}$	Isp	$T=v_{ex}*R$
Reaction Control System, RCS	m_{fuel}	Frame	Burn Duration	(Hypergolic)	Exhaust Velocity	s	Thrust
	(kg)	(kg)	(s)		(m/s)	(s)	(kN)
Command Module		5560	Pulsed				
Service Module			Pulsed	MMH/N2O4			91
RCS	120		Pulsed	Aerozine/N2O4		290	3.87
Lunar Excursion Module							
Ascent RCS	287	4700	Pulsed	Aerozine/N2O4		311	10*410N
Ascent APS	311	2353	Pulsed	Aerozine/N2O4		311	16
LM Descent - DPS	8200		30	Aerozine/N2O4		311	45

1,050 kg Reaction Control System, RCS

CSM: Command Module, CM and Service Module, SM

The CM is the cabin that houses the three astronauts. It sits ontop of the Service Module and contains the Guidance and Navigation computer controls, which are operated by the pilot. The Command Module's Attitude Control System contains the Service Propulsion System and Reaction Control System. SM.The SM contains the SPS and RCS engines. The CM's mass is 12,250 lb (5,560 kg).
The SM consisted of twelve 93-pound-force (410 N) attitude control jets; ten were located in the aft compartment, and two pitch motors in the forward compartment. Four tanks stored 270 pounds (120 kg) of hypergolic monomethylhydrazine fuel and nitrogen tetroxide oxidizer (MMH/N2O4). The system produced small pulses or bursts thrusts as needed over a 30 minute mission period.

Service Propulsion System is the Main SM Thrust Engine

The SPS engine was used to place the Apollo spacecraft both into and out of lunar orbit, and for mid-course corrections between the Earth and Moon. It also served as a retrorocket to perform the deorbit burn for Earth orbital Apollo flights. The engine selected was the AJ10-137,[9] which used Aerozine 50 (50:50 mix by weight of hydrazine and unsymmetrical dimethylhydrazine) as fuel and nitrogen tetroxide (N2O4) as oxidizer to produce 20,500 lbf (91 kN) of thrust. It needed sufficient fuel to both get it down to the moon's surface and back up.

The Reaction Control System (RCS) provides Rotation Control in All Three Axes

The CS provides the thrust to control spacecraft rates and rotation in all three axes in addition to any minor translation maneuvers. From an entry interface of 400,000 feet, the orbiter is controlled in roll, pitch and yaw axes with the aft RCS thrusters.
The forward RCS has 14 primary and two vernier engines. The aft RCS has 12 primary and two vernier engines in each pod. The primary RCS engines provide 870 pounds of vacuum thrust each, and the vernier RCS engines provide 24 pounds of vacuum thrust each.The RCS pulses placed the CM in its proper position for re-entry into the Earth's atmosphere.
RCS propellant mass: 270 lb (120 kg) RCS engine mass: twelve x 73.3 lb (33.2 kg)

$$870\text{lbf} = 3.87 \times 10^3\,\text{N}$$

Lunar Excursion Module, LEM or LM: "Eagle"

LEM was the lander portion of the Apollo spacecraft built for the US Apollo program by Grumman Aircraft to carry a crew of two from lunar orbit to the surface and back. Designed for lunar orbit rendezvous, it consisted of an ascent stage and descent stage, and was ferried to lunar orbit by its companion Command/Service Module (CSM), a separate spacecraft of approximately twice its mass, which also took the astronauts home to Earth.The LEM was carried above the CM into space. After detaching and completing its mission, the LM was discarded. It was capable of operation only in outer space; structurally and aerodynamically it was incapable of flight through the Earth's atmosphere. The Lunar Module was the first manned spacecraft to operate exclusively in the airless vacuum of space. It was the first, and to date only, crewed vehicle to land anywhere beyond Earth. When ready to leave the Moon, the LM would separate the descent stage and fire the ascent engine to climb back into orbit, using the descent stage as a launch platform.

XV. Trajectory Sim of Apollo Command/Service Module (CSM) from Earth to Moon

XVI Simulation of Apollo Command/Service Module (CSM) Trajectory from Earth to Moon

This Simulation Uses the Differential Equation Solving Methodology detailed in:
arXiv:1504.07964 "***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

$kg := 1$ $m := 1$ $s := 1$ $N := 1$ $s := 1$ $min := 60s$ $hr := 3600s$ $km := 1000m$ $kmps := km$ $kph := \frac{km}{hr}$

$deg := \frac{2\pi}{360} = 0.017$ $FRAME := 999$ $t_{end} := \frac{860min}{999} \cdot FRAME + 1$

Length of Simulation

$\mathbf{t} := t_{end} = 14.3336 \cdot hr$ $mph := 0.447 \cdot 10^{-3} kmps$

$kN := 1000N$

$v_0 := 1.1km$ $G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg}$ $R_m := 1737.5km$ $CSM_0 := 16500kg$ $kgf := 9.80665N$

Atmospheric Thrust, D: Time and Force

$t_1 := 677min$ $\Delta t := 6min$ $t_2 := t_1 + \Delta t$ $t_{burn} := t_1, t_1 + \frac{t_2 - t_1}{700} .. t_2$ $D_r := 34kN$ $D(t) := if(t_1 < t < t_2, D_r, 0kgf)$

Define Gravitational and Dynamics Equations for Earth and CSM

$$\begin{pmatrix} m_m & x_{0m} & y_{0m} & x'_{0m} & y'_{0m} \\ m & x_0 & y_0 & x'_0 & y'_0 \end{pmatrix} := \begin{pmatrix} 7.35 \cdot 10^{22} \ kg & 0 \ m & 0 \ m & 0 \ kph & 0 \ kph \\ 16500kg & -50000 \ km & R_m + 3000km & v_0 & 0kph \end{pmatrix}$$

$t_2 = 683 \cdot min$ $t_{end} = 5.16 \times 10^4$ $CSM(t) := CSM_0$ $D_r = 3.4 \times 10^4$

Given

Set of Differential Guidance Equations for CSM

$\mathbf{x}(0s) = \mathbf{x_0}$ $\mathbf{v_x}(0s) = \mathbf{x'_0}$

$$CSM(t) \cdot \left(\frac{d}{dt} \mathbf{v_x}(t)\right) = -G \cdot \frac{CSM(t) \cdot \mathbf{m_m} \cdot \mathbf{x}(t)}{\left(\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}\right)^3} - \frac{D(t) \cdot \mathbf{y}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}}$$

$\mathbf{v_x}(t) = \mathbf{x}'(t)$

$\mathbf{y}(0s) = \mathbf{y_0}$ $\mathbf{v_y}(0s) = \mathbf{y'_0}$

$$CSM(t) \cdot \left(\frac{d}{dt} \mathbf{v_y}(t)\right) = -G \cdot \frac{CSM(t) \cdot \mathbf{m_m} \cdot \mathbf{y}(t)}{\left(\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}\right)^3} + \frac{D(t) \cdot \mathbf{x}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}}$$

$\mathbf{v_y}(t) = \mathbf{y}'(t)$

Differential Equation Solver

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{v_x} \\ \mathbf{v_y} \end{pmatrix} := Odesolve\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{v_x} \\ \mathbf{v_y} \end{pmatrix}, t, t_{end}\right]$$

$t := 0s, \frac{t_{end}}{7000} .. t_{end}$ $t_{spdn} := 0s, \frac{t_2}{7000} .. t_2$ $\alpha := 0, \frac{2\pi}{1000} .. 2\pi$

$t_{orb} := t_2, t_2 + \frac{t_2 - t_1}{70} .. t_{end}$ $\mathbf{D} := D(t_{end})$ $R_{LM}(t) := \sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}$

$D_x(t) := \frac{D(t) \cdot \mathbf{y}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}}$ $D_y(t) := \frac{D(t) \cdot \mathbf{x}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}}$

$m_s := CSM(t_{end})$

$Y_{dot} := \left(\mathbf{y}(t_2 + 40s) \quad \mathbf{y}(t_{end})\right)^T$

$F_x := -D_x(t_1 + 10s)$ $F_y := -D_y(t_1 + 10s)$ $F_y = -1.082 \times 10^4 \cdot kg$ $\mathbf{D} = 0 \cdot kgf$ $F_x = -3286.6 \cdot kgf$ $X_{dot} := \left(\mathbf{x}(t_2 + 40s) \quad \mathbf{x}(t_{end})\right)^T$

Simulation of Apollo Command/Service Module Trajectory from Earth to Lunar Orbit

Apollo 11 CS< approached the moon.

Descent speed (s) is 1.69 km/s. Sim is a good match.

Results of Simulation for Lunar Height and Speed

See Red Band in Below Plot for Lunar Capture where. 2 burns: Retro-thrust to slow to 2.5km/s. 1st burn:6 minutes into 315 x 111 km orbit. 2nsd:17 sec to circulize orbit to 69 miles. 2 more corrections.

$h(t) := R_{LM}(t) - R_m$

$R_{LM}(t_1) = 2376 \cdot km$ $\quad h(t_1) = 638.8 \cdot km$ $\quad h(t_2) = 483.521 \cdot km$

$R_{LM}(t_2) = 2221 \cdot km$ $\quad s(t_1) = 2.258 \cdot kmps$ $\quad s(t_2) = 1.627 \cdot kmps$

$s(t) := \sqrt{v_x(t)^2 + v_y(t)^2}$ $\quad v_{escape} = 2.4\ kmps$

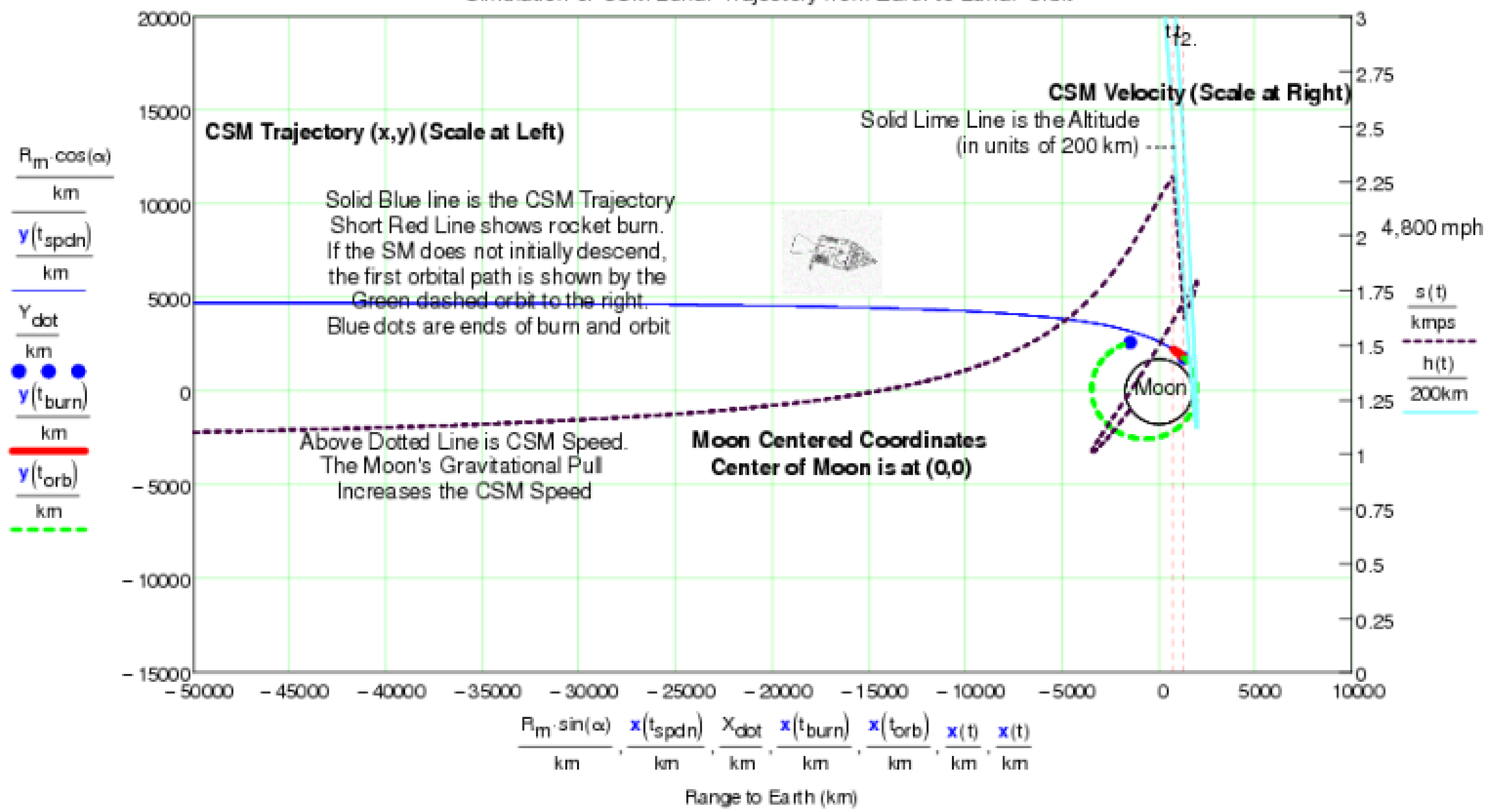
$s(t_2) = 1.627 \cdot kmps$ $\quad s(t_{end}) = 1.205 \cdot kmps$

$s(t_2) = 3.641 \times 10^3 \cdot mph$ $\quad s(t_{end}) = 2.695 \times 10^3 \cdot mph$

$x(t_1) = 735.841 \cdot km$ $\quad x(t_2) = 1.276 \times 10^3 \cdot km$

$\frac{t_1}{min} = 677$

Simulation of CSM Lunar Trajectory from Earth to Lunar Orbit



XVI: Simulation of Descent from Orbit to Moon Surface, Lunar Orbit Descent, LOD

Day 4: Descent has four stages: Orbital, Braking, Approach, and finally, Hovering/Landing

Braking Phase Descent began at 48,814 feet (15 km) from a slightly elliptical coasting lunar orbit, and time to touchdown is 12 minutes. (This would require an average velocity of about 20.5 m/s or 46 mph). Descent initial orbit velocity is 1.695 km/s (5560 fps). Attitude goes from 24º to 42º in 8 minutes. Target Location is the Sea of Tranquility is centered at latitude II. 5' N and longitude 23. 5° E. **See Section XVII for Nav. and Guidance**

Descent is computer controlled. 4 radars target landing site and compute the time to go from current to desired conditions, TGO. A Quadratic Law is used to computed TGO from measured jerk, velocity, and altitude using the NAV routines. The acceleration differential between commanded and lunar acceleration is calculated. This is converted to required thrust and throttle time. When throttle time exceeds throttle region of 10 to 60%, full throttle or thrust is applied. The Ignition logic determines the time for burn to be applied. Guidance logic is used to steer craft to selected landing location. target location determines the required trajectory shaping.

Descent uses two Control Factors: Uses Two/Throttle and Pitch Angle - **See Throttle in Graph Below**

Approach Phase. During the approach phase, the altitude decreases from 7000 to 500 feet, the range decreases from approximately 4 5 nautical miles to 500 feet, and time of flight is approximately 1 minute 40 secs.

LEM (Eagle) Descent Vehicle and Engine Parameters

Mass of LEM $m_{lod} := 8200\text{kg}$

Thrust of LEM $T_{lod} := 44.5\text{kN}$

$g_{moon} := 1.62\frac{\text{m}}{\text{s}^2}$

Initial Orbital Velocity $v_{orb} := 1.695\frac{\text{km}}{\text{s}}$

Throttle Profile%: $\text{Throttle}_{des}(t) := \text{if}\left[t < 60, 0.4, \text{if}\left[t < 400, 1, \text{if}\left[t < 500, 0.6, 0.6\cdot e^{-1\cdot\left(\frac{t-500}{400}\right)}\right]\right]\right]$

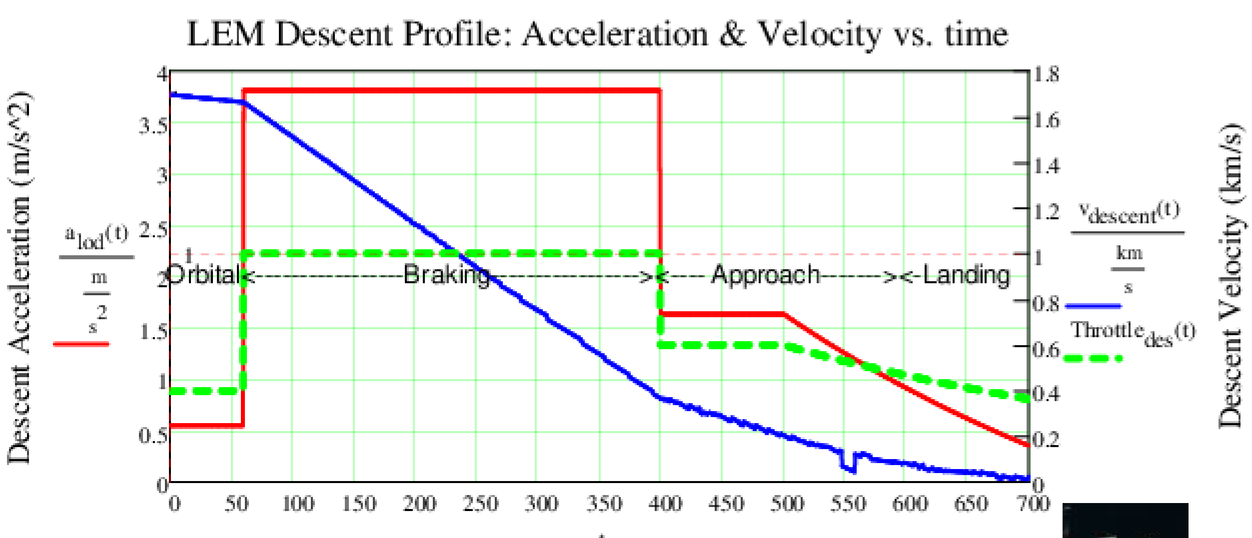
Full Throttle: $\frac{T_{lod}}{m_{lod}} - g_{moon} = 3.807\frac{\text{m}}{\text{s}^2}$

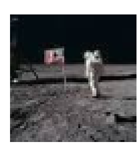
LEM Acceleration: $a_{lod}(t) := \text{Throttle}_{des}(t)\cdot\frac{T_{lod}}{m_{lod}} - g_{moon}$

Simulate Descent Velocity: $v_{descent}(t) := v_{orb} - \int_0^t a_{lod}(t)\, dt\cdot s$

Day 4: Simulation of LEM "Eagle" Descent to Moon Surface

Green Dotted Line -Throttle Profile
Red: Acceleration
Blue: Orbital to Descent Velocity





July 20, 1969, 20:17:14 UTC: "***Houston, Tranquillity Base here. The Eagle has landed.***"

XVII: Simulation of Ascent from Moon Surface to Orbit, Lunar Orbit Ascent, LOA

LEM Ascent Stage - Day 5 From Tranquility Base

Ascent has a single objective, namely, to achieve a satisfactory orbit from which rendezvous with the orbiting GEM can subsequently be performed. Nominally, insertion into a 9 by 45 mile at an altitude of 60,000 feet, is desired.
Ascent Engine Throttling Notes: The LMDE operates in two regimes: Full Throttle Power (FTP) is approximately 94.2% of rated thrust (9,900 lbs), and the Throttling Regime goes from 12.2% of rated thrust to 65%~ rated thrust (1,280 lbs to 6,825 lbs).

the Lunar Module Ascent stage didn't't need to attain the 2.4 km/s to escape the Moon's gravity, it just had to reach a lunar orbit (orbital speed in the range **1.5-1.7 km/s**) to rendezvous with the CSM. **Seven minutes** after ignition the astronauts would be in lunar orbit awaiting the rendezvous with the CSM. Once the LM crew transferred into the CSM, their LM Ascent stage was abandoned. All three crew returned to Earth in the Command Module.

The engines of the RCS are placed at the four corners of the lunar module because the center of gravity is shifted from the axis of the ascent module as the tanks are emptying. Thus the combined forces of the thrust and the lunar gravity create a torque which makes the ascent module turn clockwise. So that the ascent module keeps a steady direction, this torque needs to be corrected by applying a counter torque. The powered ascent is divided into two operational phases: vertical rise and orbital insertion.
5536 ft/s at 60,000 ft. The required Δv is 6056 fps. For Apollo 14 the theoretical minimum would have been 6045.3 fps, **1842.6 m/s over ~430 seconds** of flight time. After 10 sec of Thrust Angle from vertical of 0 degrees thrust angle varies linearly from 50 to 90 degree from 10 to 430 s.

Run Simulation with Variable Pitch Angle and Engine Throttled from 90 to 64% Thrust

$5536\frac{ft}{s} = 1.687\times 10^3\frac{m}{s}$ $\qquad \theta_{asc}(t) := \frac{\pi}{2}\cdot\left[1-\Phi(t-10)\cdot\left(0.31+\frac{t}{700}\right)\right]$ $\qquad g_{thr\_asc}(t) := g_{moon}\cdot\sin(\theta_{asc}(t))$

$m_{loa} := 2132kg$ $\qquad T_{loa} := 16kN$ $\qquad a_{loa}(t) := \frac{T_{loa}}{m_{loa}} - g_{thr\_asc}(t)$ $\qquad a_{loa}(450) = 7.385\frac{m}{s^2}$

$Throttle_{asc}(t) := if(t< 40, 0.9, if(t< 435, 0.648, 0.75))$ $\qquad Thr_{asc}(t) := 100\,Throttle_{asc}(t)\,3500lbf = 1.557\times 10^4 N$

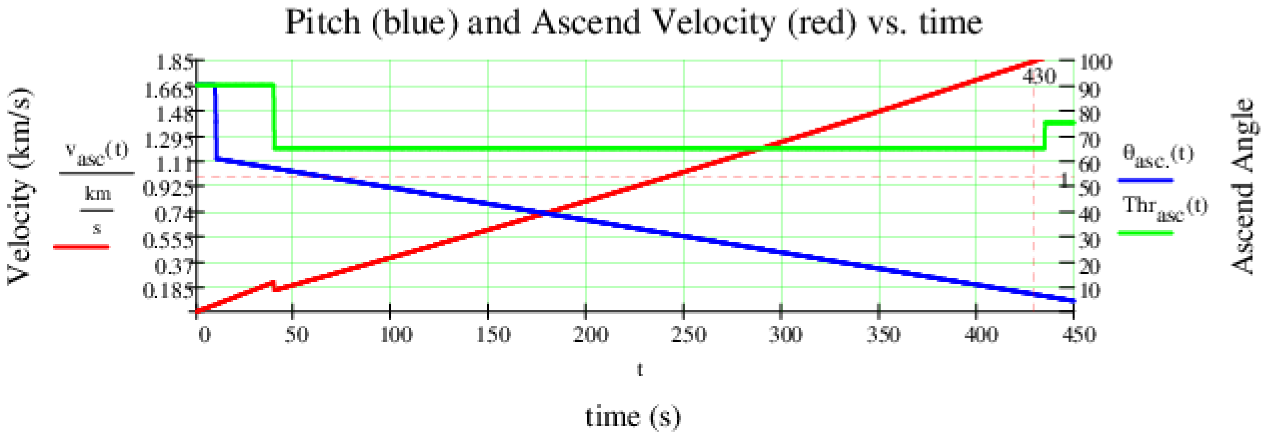
$\theta_{asc.}(t) := 90\cdot\left[1-\Phi(t-10)\cdot\left(0.31+\frac{t}{700}\right)\right]$

Ascent Target 1843 m/s at 430 s

Simulate Ascent Velocity: $\qquad v_{asc}(t) := Throttle_{asc}(t)\cdot\int_0^t a_{loa}(t)\,dt\cdot s$ $\qquad v_{asc}(430) = 1.844\times 10^3\frac{m}{s}$

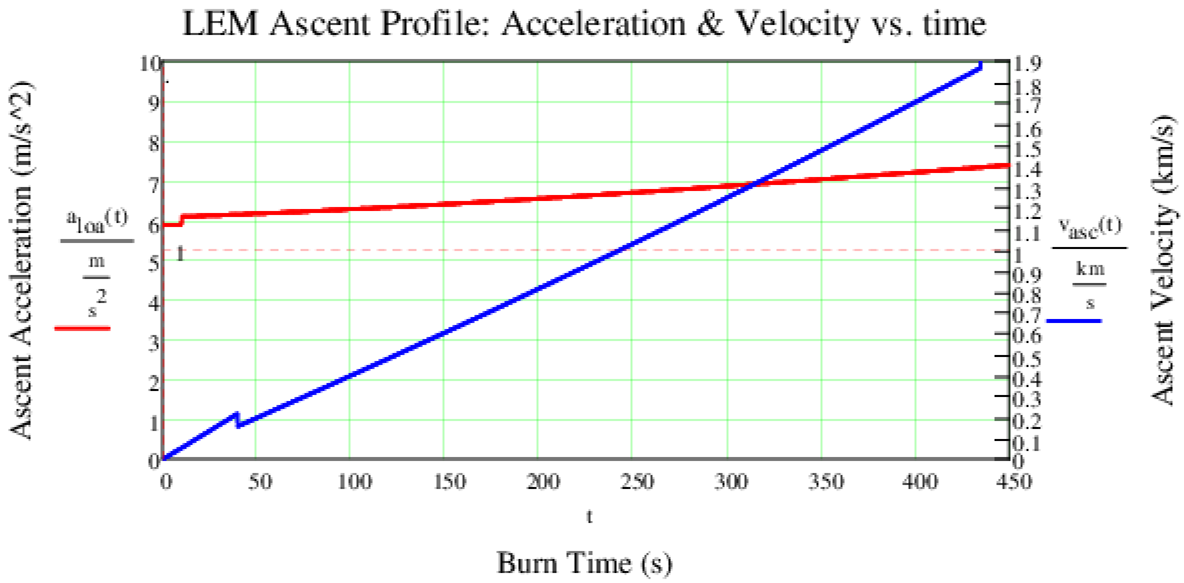
LEM Ascent to Lunar Orbit Simulation:

Blue -Thrust Angle Profile of Trajectory
Red: Ascent Velocity Rise Up to Orbital Velocity
Green: Engine Throttle



Plot of LEM's Ascent from Moon's Surface to Lunar Orbit:

Red: Ascent Acceleration
Blue: Ascent to Lunar Orbital Velocity



Day 6: Rendezvous and Docking

Eagle is now safely back in its initial lunar orbit.

https://history.nasa.gov/afj/ap11fj/19day6-rendezvs-dock.html

To rendezvous with his CSM the LEM executed a series of burns by its reaction control thrusters, controlled by the LEM computer on the basis of data supplied by Houston mission control, that initially put it in a circular orbit at an altitude of 69 miles concentric with the CSM, and then slowed it down to dock with the CSM. The LEM commander, took over control for the final docking maneuver. Return docking was very crucial and difficult. The crew returned to the command module and the hatch was sealed. The CSM and LM separated and the lunar module was jettisoned. Only the LEM ascent stage was left in lunar orbit.

XVIII: Command Service Module "Columbia" Return Flight to Earth Orbit

Trans-Earth Injection (TEI) by the SPS engine on the CSM

The Eagle reached an initial orbit of 11 by 55 miles above the moon, and when Columbia was on its 25th revolution. As the ascent stage reached apolune at 125 hours, 19 minutes, the reaction control system, or RCS, fired so as to nearly circularize the Eagle orbit at about 56 miles, some 13 miles below and slightly behind Columbia. The SPS fired for two-and-a-half minutes when Columbia was behind the moon in its 59th hour of lunar orbit.

A Trans-Earth Injection (TEI) is a propulsion maneuver used to set a spacecraft on a trajectory which will intersect the Earth's Sphere of influence, usually putting the spacecraft on a **Free Fall** return trajectory to earth. It was performed by the SPS engine on the Service Module after the undocking of the (LM) Lunar Module if provided. An Apollo TEI burn lasted approximately 203.7 seconds, providing a postgrade velocity increase of **1,076 m/s** (3,531 ft/s).

Sim: SPS engine Burn on the Service Module after the undocking from the Lunar Module (LM)

TEI Control Parameters from Influence of Moon (g-moon) to Earth (g-earth)

SM Thrust — $T_{SM} := 91kN$

Throttle Factor — $Throttle := 0.9$

SM acceleration — $a_{moon} := \frac{Throttle \cdot T_{SM}}{m_{SM}} - \sin(20deg) \cdot g_{moon}$

SM mass — $m_{SM} := 14000kg$

$t_{burnSM} := 204s$

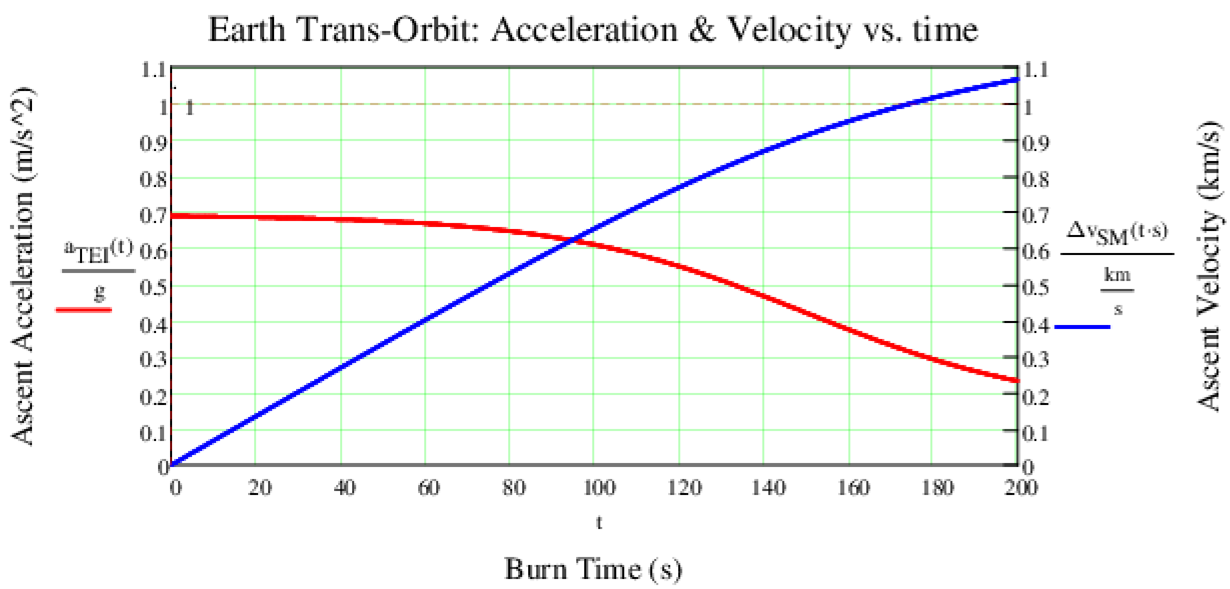
$a_{moon} = 3.269 \cdot g_{moon}$

$$a_{TEI}(t) := \left[\frac{a_{moon}}{1 + e^{-\left(\frac{150-t}{29}\right)}} + .151g \right]$$

$$\Delta v_{SM}(t) := \int_0^t a_{TEI}\left(\frac{t}{s}\right) dt$$

Apollo Mission Data = 1.076 km/s

$\Delta v_{SM}(t_{burnSM}) = 1.075 \times 10^3 \frac{m}{s}$



XIX. Trans-Earth Coast, Mid-Course Correction, and CM/LEM Separation

Trans Earth Coast.
Apollo 11's return voyage to the Earth was powered by the Earth's gravity. The gravitational effects of the trans-lunar coast were reversed. The velocity of the CSM initially slowed due to the gravitational pull from the earth, but as the spacecraft moved away from the moon, the moon's gravitational effect diminished, while the Earth's gravitational pull increased. Once the earth's gravitation became dominant, the space craft was **accelerated into a freefall,** all the way to the Earth, reaching a velocity of 25,000 mph, as it entered the Earth's atmosphere over three days later.

Command Module/Service Module Separation.
Shortly before entering the Earth's atmosphere at an altitude of 400,000 feet (75 miles), the service module was jettisoned by simultaneously firing of the reaction control thrusters in both the service module and the command module. The command module is then rotated by a 180° to turn the blunt end toward the Earth.

Reentry.
Landing is just as dangerous as taking off, and ***the precise reentry trajectory is critical*** to making a safe landing. Once initiated there is ***no possibility of a second chance*** by going around and trying again if things go wrong. The initial atmospheric drag of 0.05 g experiences by the capsule as it entered the atmosphere, triggers the Earth landing subsystem, which controlled the reentry process. The Command Module entered the atmosphere, blunt end first, at 400,000 feet with a velocity approaching 25,000 mph at an angle of 6.488° to the horizontal, and flew about 1240 miles from the Earth to the designated landing point in the Pacific ocean. At the very high entry velocity, it compressed the air as the capsule heated its surface to up to around 2760° C, or 5000F, hot enough to vaporize most metals, turning the capsule into a shooting star. A 2 1/2 inch thick, sacrificial ablative heat shield, which burns and a erodes, in a the self-controlled way, carrying heat away with its combustion products, protected the capsule from the heat of reentry.

Mid-Course Correction.
Similar to the outward journey, the velocity and angle of approach to the Earth had to be very precisely controlled to ensure capture by the Earth's gravity and splashdown into the designated area

XX. Re-entry into the Earth's Atmosphere:

Challenges of High Speed Atmospheric Entry

$\text{Mega} := 10^6$

Crewed space vehicles must be slowed to subsonic speeds before parachutes or air brakes may be deployed. Such vehicles have kinetic energies typically between 50 and 1,800 megajoules (Apollo 50,000 megajoules), and atmospheric dissipation is the only way of expending the kinetic energy. The amount of rocket fuel required to slow the vehicle would be nearly equal to the amount used to accelerate it initially, and it is thus highly impractical to use retro rockets for the entire Earth re-entry procedure. While the high temperature generated at the surface of the heat shield is due to adiabatic compression, the vehicle's kinetic energy is ultimately lost to gas friction (viscosity) after the vehicle has passed by. Ballistic warheads and expendable vehicles do not require slowing at re-entry, and in fact, are made streamlined so as to maintain their speed. For Earth, atmospheric entry occurs at the Kármán line at an altitude of 100 km (62.14 mi / ~ 54 nautical mi) above the surface.

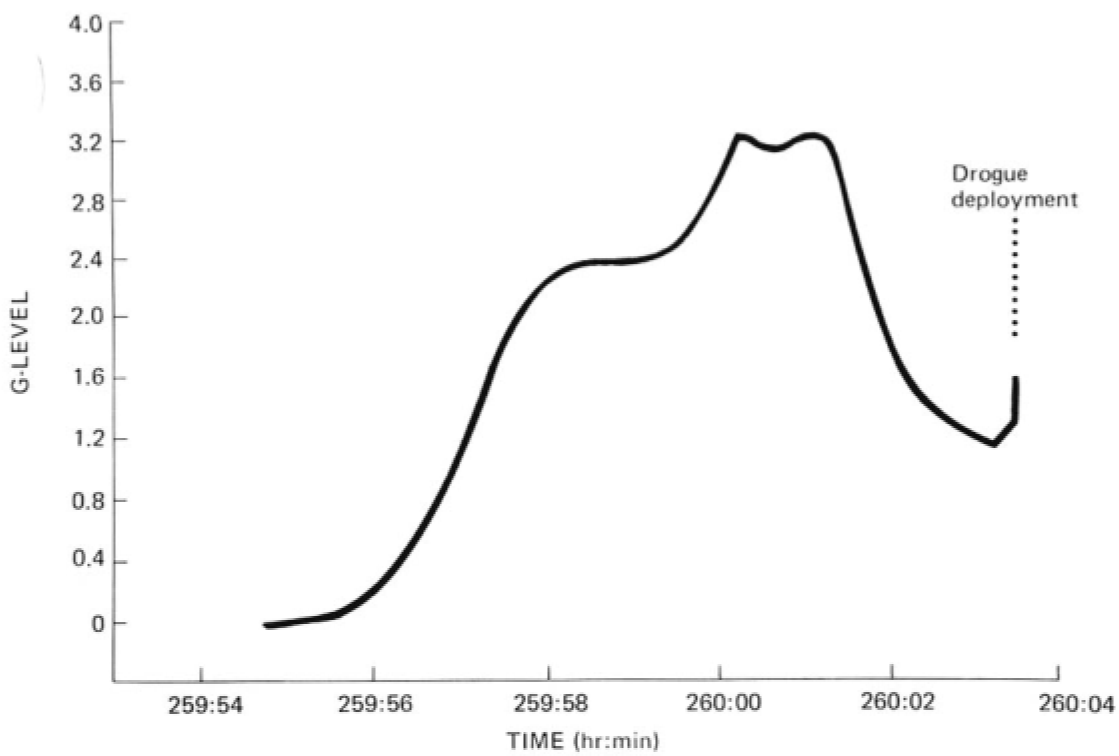
Apollo 11 Re-entry procedures were initiated July 24, 44 hours after leaving lunar orbit. The SM separated from the CM, which was re-oriented to a heat-shield-forward position.

Apollo 11 was 19,914 nautical miles [36,881 km] from the Earth approaching at a velocity of 13,695 feet per second [**4,174 m/s or Mach 12.3**]. It was 2 hours, 14 minutes, 16 seconds away from entry into the atmosphere.
The CM then adjusted its attitude -- or orientation respective to the Earth's surface -- using its thrusters so that the base of the module faced towards the Earth's surface.

Apollo Journal: https://www.hq.nasa.gov/alsj/

Earth Orbital Re-entry Atmospheric Drag Profile - Apollo 7

SP-368 Biomedical Results of Apollo



XXI. Strategies for Dissipation of Heat from Atmospheric Braking

Command Module, CM, Heat Shields

The interior of the command module must be protected from the extremes of environment that will be encountered during a mission. The heat of launch is absorbed principally through the boost protective cover, a fiberglass structure covered with cork which fits over the command module like a glove. The boost protective cover weighs about 700 pounds and varies in thickness from about 3/10 of an inch to about 7/8" (at the top).

There are 4 possible ways to dissipate or shield a rocket's interior from this heat.
1. ICBMs come in at high angles and spend little time in that atmosphere to minimize random effects of turbulence and distortions of the atmosphere and thus attain high target accuracy.The more time a missile spends in the atmosphere, the less accurate.it will be. ICBMs employ heat sinks, which delays the time it takes to spread the heat. This thermal delay to peak temperature, protects the electronics before the target hit.

2. The Space Shuttle came in at an angle and, slowed down, and used tiles that get very hot & radiated the heat away. It used 22,000 ablative insulated silica glass tiles, glued to the aluminum body with silicone adhesive, that heated up to very high temperatures, but provided thermal insulation, and then radiated the heat away. The risk of this method was that loss of just a few tiles in a critical area, left that region unprotected, and the result could be catastrophic. The result could be like a blow torch on the aluminum body. This was the cause of the Columbia disaster in 1981. It was estimated that tiles represented 10% of the risk of failure of Space Shuttle.

3. Apollo Missions used the ablation method. They used flat nose cones and a shallow entry angle to slow down entry. The slow re-entry also minimized the de-acceleration to protect the astronauts.
The flat Apollo CM nose cone heated up, but they had a ablative material deposited on the cones that absorbed and ablated the heat. This heat went into ablating (melting and boiling off) the nose cone material. The process was similar to pouring water on a fire, or using ice to cool down a drink on a hot day, only instead of melting of ice in drinks, the ablative material vaporizes. Vaporization, like boiling water, absorbs and carries huge amounts of heat away. Cone materials are used that have a very large heat of fusion. The ablative material that did this job for Apollo was a silica impregnated phenolic epoxy resin, a type of reinforced plastic.
Total weight of the shield was about **3,000 lbs. Heat of fusion** 143 kCal/mol. 143 kCal/68 gm = **8.8*10^6 J/kg**

The temperature on the **CM's surface climbed up to 5,000 degrees Fahrenheit**, but the heat shields protected the inner structure of the CM. The heat shield was ablative, which means that it was designed to melt and erode away from the CM as it heated up. From the ground, it would look as if the CM had caught on fire during its descent. In reality, the ablative covering is what kept the astronauts inside the CM safe -- the material diverted heat away as it vaporized.

Calculate Terminal Velocity of Command Module Prior to Parachute Deployment, v_{term}

Mass of the command module, CM: 5809kg
Drag coefficient, C_d, ~ 1.3 (See Apollo C_d Graph from following page). We time average this to get C_{d_avg}
Area of the CM's heat shield: 11.631 m2 (125.2 ft2)
Altitude Drogue parachutes deployed: 24,000 feet (7315.2m).
Density of the air at that altitude: 0.57 kg/m3.

$m_{CM} := 14690\text{kg} \qquad C_{d\_avg} := 1.3 \qquad A_{CM} := 11.6\text{m}^2 \qquad \rho_{air\_chute} := 0.57\frac{\text{kg}}{\text{m}^3}$

$v_{term} := \sqrt{\frac{2 \cdot m_{CM} \cdot g}{\rho_{air\_chute} \cdot A_{CM} \cdot C_{d\_avg}}} \qquad v_{term} = 183.083\frac{\text{m}}{\text{s}} \qquad v_{term} = 409.545 \cdot \text{mph}$

Energy Dissipated by Heat Shield to Atmosphere during Re-Entry

$\Delta KE(v_i, v_f) := \frac{1}{2} m_{CM} \cdot v_i^2 - \frac{1}{2} m_{CM} \cdot v_f^2 \qquad Heat\_Loss := \Delta KE\left(4.174 \cdot \frac{\text{km}}{\text{s}}, 114 \cdot \frac{\text{m}}{\text{s}}\right)$

$Heat\_Loss = 1.212 \times 10^8 \cdot \text{BTU} \qquad Heating\_Load := \frac{Heat\_Loss}{A_{CM}} \qquad Heating\_Load = 9.707 \times 10^5 \cdot \frac{\text{BTU}}{\text{ft}^2}$

$Heat\_Loss = 1.279 \times 10^5 \cdot \text{Mega} \cdot \text{J}$

This is a tremendous amount of heat that must be dissipated by the vehicle. What If the vehicle were made of solid aluminum? Al has a heat capacity of 921 J/kg-K or 0.22 BTU/lb-°F. The rise in temperature would be:

$\Delta T_{rise} := \frac{Heat\_Loss}{m_{CM} \cdot 0.22 \frac{\text{BTU}}{\text{lb} \cdot \text{F}}} \qquad \Delta T_{rise} = 1.701 \times 10^4 \, \text{F}$

The boiling point of Al is just 4,478 F.
If we did not have some way of dissipating this heat, the vehicle would melt, then and boil away.

Apollo Ablative Shield Temperature Rise

$m_{shield} := 700\text{lb} \qquad \Delta T_{rise} := \frac{Heat\_Loss}{m_{shield} \cdot \left(8.8 \cdot 10^6 \frac{\text{J}}{\text{kg} \cdot \text{K}}\right)} \qquad \Delta T_{rise} = 45.764\,\text{K}$

The atmosphere acted like a braking system on the spacecraft. To further slow the CM's descent, the spacecraft used mortar-deployed parachutes.

Parachute deployment occurred at 195 hours, 13 minutes.

The pilot chutes are deployed at about 10,000 feet (3.05 km) by a barometric switch, pulling the three main parachutes from their containers. The ELS was designed so the drogue chutes slow the descent down to roughly 200 km/h (124 mi/h) before the pilot chutes pull the main chutes, eventually slowing down the CM to 22 miles per hour (35 km/h) for splashdown and to roughly 24.5 mi/h
(**39.5 km/h**) with only two main chutes properly deployed, as it happened during the Apollo 15 splashdown.
After a flight of 195 hours, 18 minutes, 35 seconds - about 36 minutes longer than planned - Apollo 11 splashed down in the Pacific Ocean, 13 miles from the recovery ship USS Hornet.

During a water impact the CM deceleration force will vary from 12 to 40 G's. of the waves and the CM's rate of descent. A major portion of the energy (75 to 90 percent) is absorbed by the water and by deformation of the CM structure.
The module's impact attenuation system reduces the forces acting on the crew to a tolerable level. The impact attenuation system is part internal and part external. The external part consists of four crushable ribs (each about 4 inches thick and a foot in length) installed in the aft compartment.

XXII. Simulate Drag Force or Drag Coefficient on CM - Five Different Ways

We call the forward part of a space vehicle that pushes the air out of the way the nose cone. The air resistance for the nose cone is determined by a factor called the Drag Coefficient. The Drag Force is given by the relation:

$$\textbf{Drag Force or Drag} = C_d\, \rho\, v^2 A/2$$

where Cd is the drag coefficient, v is the velocity, ρ is the density of air, and A is the frontal area. The drag also depends on the angle that the nose cone makes with the vertical. It can be measured from wind tunnel testing or data from the firing of projectiles. It is considerably more difficult for the case of rocket re-entry where the density of air varies with altitude and heating causes blow torch level air temperatures.

Note from the above equation the knowing only the Drag Coefficient is not sufficient to obtain the Drag Force. We must also know how the air density. But in the case of a rocket, the air density changes significantly with altitude. Thus we also need to know how the altitude changes.But the altitudes changes with both time and velocity. The density also changes with air boundary layer temperature, which is a function of heatflow to CM.

<u>We will get values of the Drag Force by 5 different methods:</u>

First we must model the variation of air density with respect to altitude

0. Atmospheric Variation of Altitude (z) of Temp(z), Pres(z), and ρ(z)
1. Variation of Cd with Mach Number from tables. Mach Number is the velocity in multiples of speed of sound
   From this we can create a function to approximate the variation of Cd with Mach Number.
2. Reentry Data: Velocity, Altitude, vs time Acceleration is Δv/Δt or slope. Data: *Apollo 8 Mission Report*
3.Theoretical Calculation of maximum acceleration. This does not model the altitude.
4. Constant Cd with variation of density ρ with altitude Obtain altitude from velocity or time data from method 2
5. Constant Cd with constant value of density ρ

<u>The method that is ultimately successful is #2, that is,determine the acceleration and then simulate it.</u>

0. Atmospheric Variation with Altitude (z) of Temp(z), Pres(z), and ρ(z)

Atmosphere model: US 1976 Standard Atmosphere (z < 11,000 m): Troposphere

The ideal gas law in molar form, which relates pressure, density, and temperature:

$$P = \rho R_{\text{specific}} T$$

Temperature (K) -> $T(z) := 288.149 - .00649 \cdot \frac{z}{m}$

Pressure (Pa) -> $p(z) := 101325 \cdot \left(1 - 2.26 \cdot 10^{-5} \cdot \frac{z}{m}\right)^{5.25}$

Density (kg/m³) -> $\rho(z) := \frac{p(z \cdot 10^3)}{T(z \cdot 10^3) \cdot 287} \cdot \frac{kg}{m^3}$

Speed of Sound (m/sec) $v_s(z) := \sqrt{401.8 \cdot T(z)}$

$\rho(1ft) = 1.19 \frac{kg}{m^3}$

1. Variation of Drag Coefficient with Velocity (Mach 0 to 10):

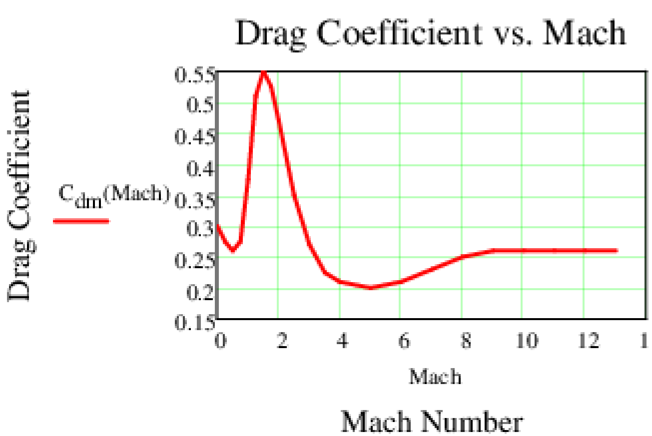
$\text{Mach1} := 340 \frac{m}{s}$

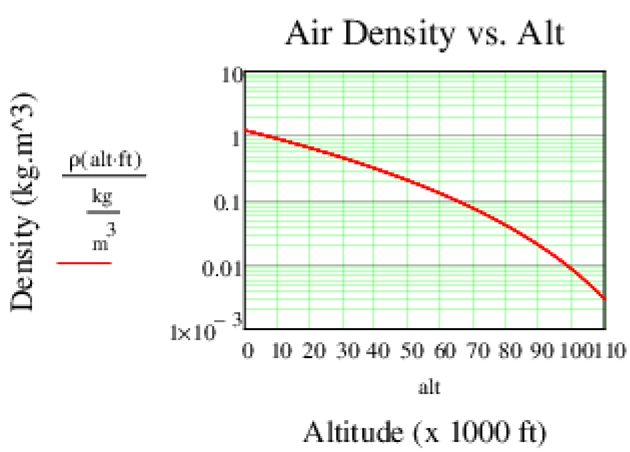
$\text{Mach1} = 760.558 \cdot mph$

$\text{Mach} := (0\ \ 0.25\ \ 0.5\ \ 0.75\ \ 1\ \ 1.25\ \ 1.5\ \ 1.75\ \ 2\ \ 2.5\ \ 3\ \ 3.5\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13)^T$

$C_d := (0.3\ \ 0.275\ \ .26\ \ .275\ \ .375\ \ .51\ \ .55\ \ .525\ \ .47\ \ .35\ \ .27\ \ .225\ \ .21\ \ .2\ \ .21\ \ .23\ \ .25\ \ .26\ \ .26\ \ .26\ \ .26\ \ .26)^T$

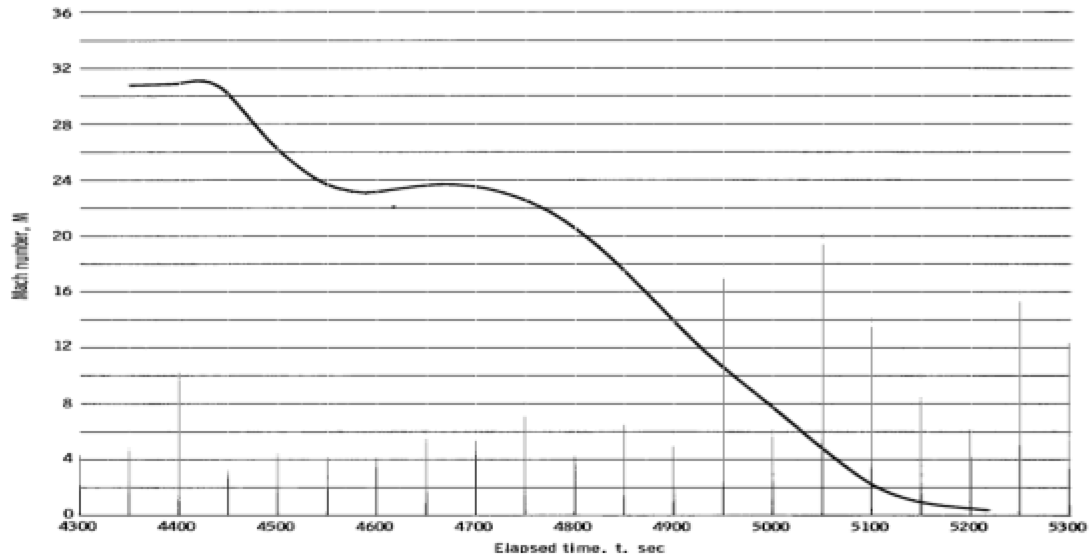
Interpolate for Altitude vs Mach # -> $C_{dm}(M) := \text{interp}(\text{cspline}(\text{Mach}, C_d), \text{Mach}, C_d, M)$ $\quad C_{dm}(13) = 0.26$



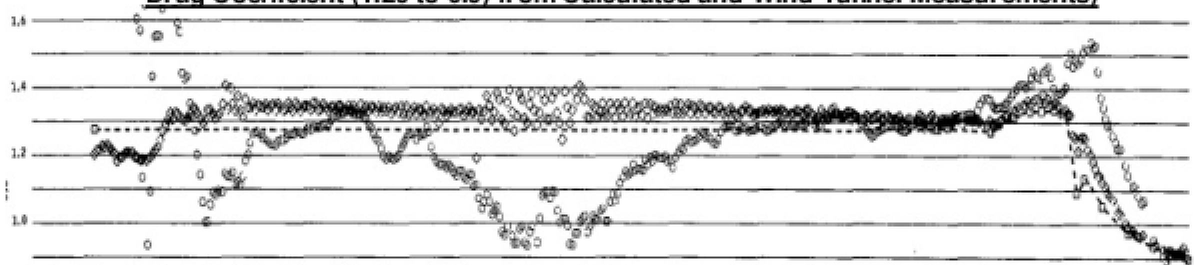


XXIII. Apollo Re- Entry: Velocity, Altitude, CD Flight Data vs. time

Apollo Time History of Trajectory Parameters AS-202

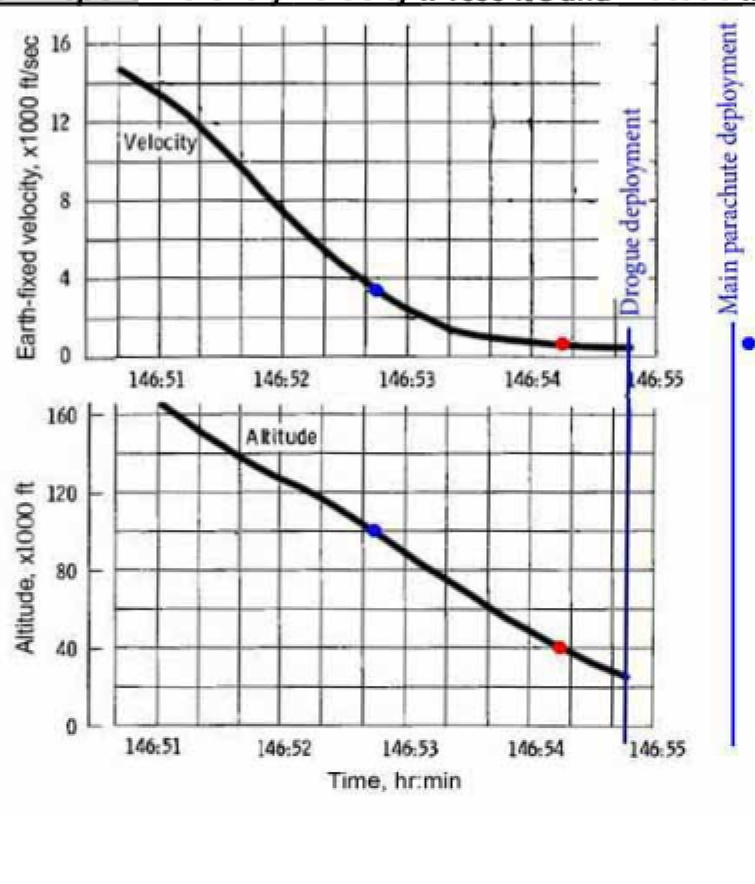


Drag Coefficient (1.29 to 0.9) from Calculated and Wind Tunnel Measurements)



Elapsed time from 4300 to 5220 seconds

2. Apollo 8 Mission Report: Re-entry Velocity x 1000 ft/s and Altitude x 1000 ft vs. time



***2. Apollo 8 Mission Report*, pg 5-22: Re-entry Altitude x 1000 ft vs Velocity x 1000 ft/s**

Put Data in Functional Form: **Alt(v) & Vel(t)**. Mach Number Velocity Decreased from 31 to 0

$t_{dn} := (0\ \ 20\ \ 39\ \ 60\ \ 80\ \ 100\ \ 118\ \ 128\ \ 130\ \ 150\ \ 160\ \ 180\ \ 190\ \ 200\ \ 220\ \ 240)^T$

Smooth Vel & Alt Data

$vel_{dn} := (14.8\ \ 13.5\ \ 12\ \ 10\ \ 8\ \ 5.8\ \ 4\ \ 3.5\ \ 3\ \ 2.8\ \ 1.8\ \ 1.7\ \ 1\ \ .8\ \ 0.79\ \ 0.7)^T$ $\quad vel_{dns} := \text{ksmooth}(t_{dn}, vel_{dn}, 40)$

$alt_{dn} := (175\ \ 160\ \ 158\ \ 140\ \ 130\ \ 120\ \ 105\ \ 100\ \ 97\ \ 90\ \ 78\ \ 60\ \ 58\ \ 50\ \ 38\ \ 28$ $\quad alt_{dns} := \text{ksmooth}(t_{dn}, alt_{dn}, 40)$

Reverse Order, Make data ascending for Interp Fn $\quad n := 0, 1 .. 15 \quad veldni_n := vel_{dn_{15-n}} \quad altdni_n := alt_{dn_{15-n}}$

Interpolate Data for Altitude vs Velocity: $\quad Alt_{dni}(V) := \text{interp}(\text{cspline}(vel_{dni}, alt_{dni}), vel_{dni}, alt_{dni}, V)$

Add Units of fps $\quad fps := ft \cdot sec^{-1} \quad Alt_{dn}(v) := Alt_{dni}\left(\frac{v}{10^3 fps}\right) 10^3 ft \quad Alt_{dn}(10^3 fps) = 1.768 \times 10^4 m$

2. Break Velocity Data, vel, into 2 Regions of different Slopes or Accels - Below Graph Shows 2 Slopes

This Mathcad "slope" function finds the slope

$$accel1 := \text{slope}(\text{submatrix}(t_{dn}, 0, 6, 0, 0), \text{submatrix}(vel_{dn}, 0, 6, 0, 0)) \cdot \frac{1000ft}{s^2}$$

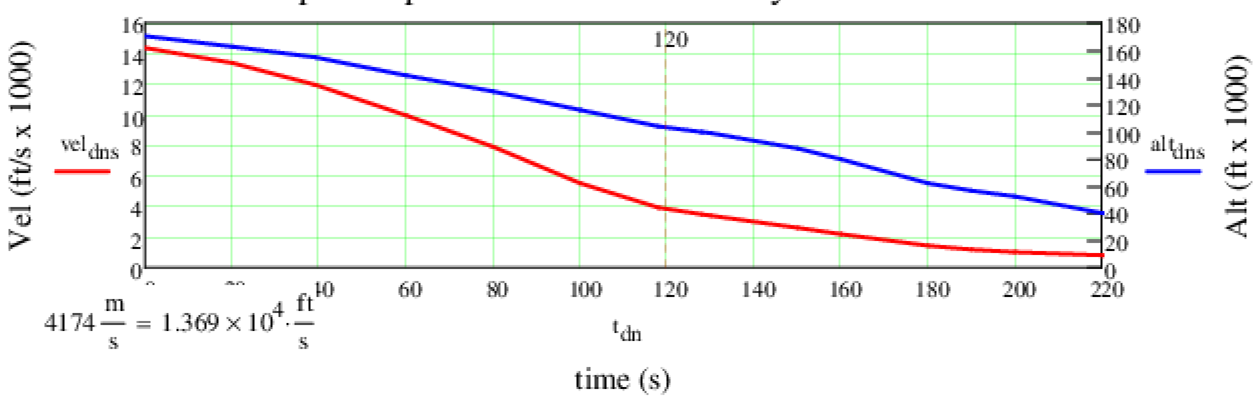
$$accel2 := \text{slope}(\text{submatrix}(t_{dn}, 7, 13, 0, 0), \text{submatrix}(vel_{dns}, 7, 13, 0, 0)) \cdot \frac{1000ft}{s^2}$$

Acceleration1 for 0 to 120s $\quad accel1 = -2.9 \cdot g$

Acceleration2 for 120 to 200s $\quad accel2 = -1.11 \cdot g$

See that Blue Velocity Curve has two slopes = Δv/Δt. Interpret this as accel1 (-2.9g) and accel2 (-1.11g)

Plot of Apollo Splashdown Data: Velocity & Alititude vs.time



$4174 \frac{m}{s} = 1.369 \times 10^4 \cdot \frac{ft}{s}$

3. Theoretical Maximum De-Acceleration from Atmospheric Drag

Parameters for the Command Module

Drag $= C_d\, \rho\, v^2 A/2 \quad \rho_{air\_avg} := 0.5 \frac{kg}{m^3}$

Earth Atmospheric Scale Altitude β - Characterizes Density Profile $\quad A_{cm} := 11.5 m^2$

$\rho0_{air} := 1.225 \frac{kg}{m^3} \quad C_d := 1.29 \quad \gamma := 50deg$

Ballistic Coefficient, BC: $\quad BC := \frac{m_{CM}}{C_d \cdot A_{cm}}$

C_d determines complex shape dependencies.

$BC = 990.226 \frac{kg}{m^2} \quad \beta := \frac{0.000139}{m}$

Maximum Acceleration from Atmospheric Drag, a_{max}: γ is the vehicles's flight-path angle

$$a_{max}(v_{re\_entry}, \gamma) := \frac{v_{re\_entry}^2 \cdot \beta \cdot \sin(\gamma)}{2e}$$

Reference: FAA Medical Studies:4.1.7
Returning from Space - Re-entry
Hi Speed Drag Calculations

$$alititude_{amax}(\gamma) := \frac{1}{\beta} \cdot \ln\left(\frac{\rho_{air\_avg}}{BC \cdot \beta \cdot \sin(\gamma)}\right)$$

3. This gives a maximum acceleration of 3.4 g. This value is close to accel from method #2.

$$a_{max}\left(3.9 \frac{km}{s}, 5deg\right) = 3.456 \cdot g$$

XXV. Atmospheric Braking & Splashdown CM Acceleration and Velocity

Simulate Re-Entry Velocity Profile for CM

$v_{entry} := 4174 \frac{m}{s}$ $\quad v_{entry} = 9.337 \times 10^{3} \cdot mph$

Spacecraft Geometry
$Diam := 12ft + 10in$
$m_{CM} := 14690kg$

Area Shield, As
$A_s := \pi \cdot \left(\frac{Diam}{2}\right)^2$

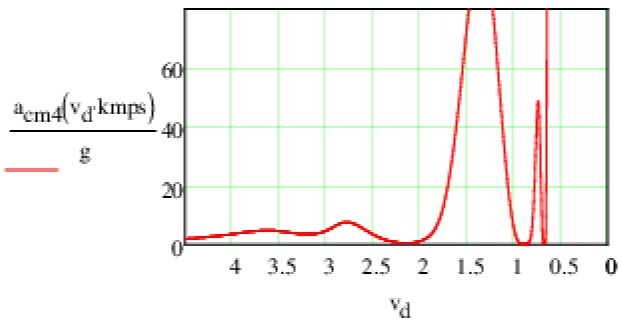
$D = C_d \; v^2 \; A/2$ $\quad mps := \frac{m}{s}$ $\quad kmps := 10^3 \frac{m}{s}$

Simulate Drag Coefficient as a Function of how Altitude - Density varies with velocity to get a(v)

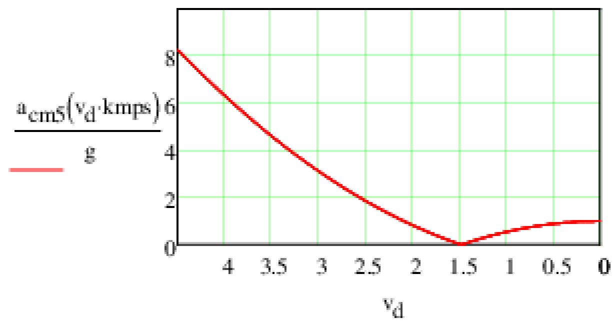
4. Drag Variable Density(Alt): $Drag(v) := \frac{1}{2} \cdot 1.29 \cdot \rho\left(Alt_{dn}\left(\frac{v}{3.3}\right) \cdot 10^{-3}\right) \cdot (v)^2 \cdot A_s$ $\quad a_{cm4}(v) := \left| g - \frac{Drag(v)}{m_{CM}} \right|$

5. Drag Fixed Altitude @ 100,000 ft : $DRAG(v) := \frac{1}{2} \cdot 1.29 \cdot \rho(100ft) \cdot v^2 \cdot A_s$ $\quad a_{cm5}(v) := \left| g - \frac{DRAG(v)}{m_{CM}} \right|$

D(v): Drag Coeff from **Density as Fn of Altitude** acm4 is not very stable. root function does not converge D(v).



5. Constant: acm5 is stable, but max g ~22 root function does not converge



5. Constant Drag Coefficient Velocity vs time

$V := 0 \cdot mps$ $\quad vel_{cm5}(t) := root\left(t \cdot sec - \int_{4000}^{V} \frac{mps}{-a_{cm5}(V \cdot mps)} dV, V\right) \cdot mps$

$vel_{cm5}(200) = \blacksquare$

Convergence tol = 0.1

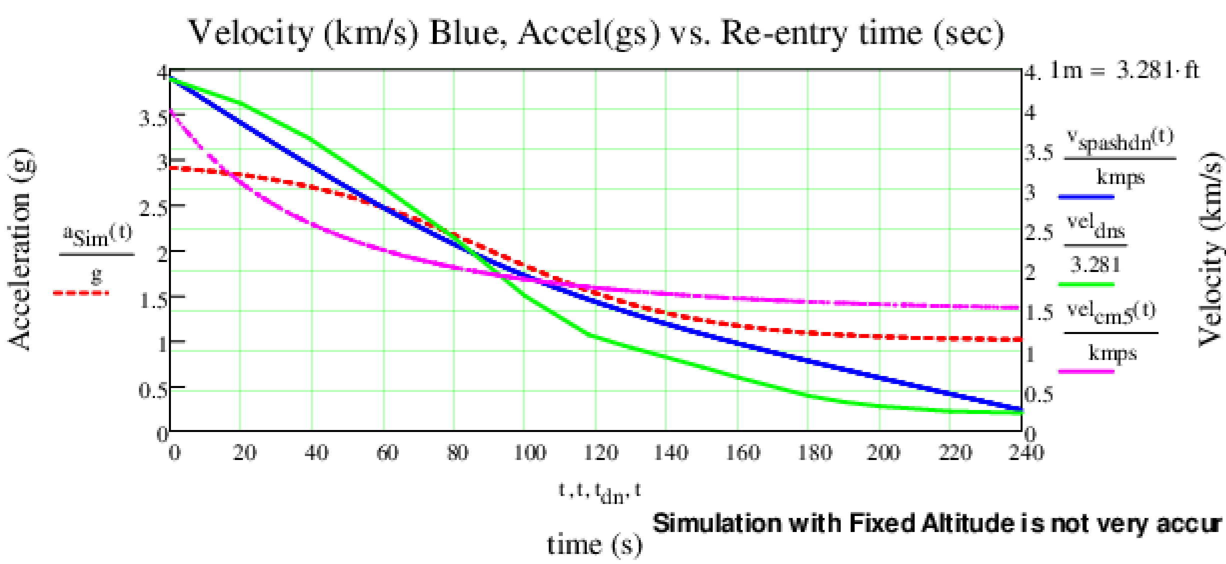
2. Simulate Splashdown Drag Data
Accel. asim, varied from 2.9 to 1.1g

$a_{Sim}(t) := \left[\frac{2}{1 + e^{-\left(\frac{90-t}{29}\right)}} + 1\right] \cdot g$

Simulation of CM Velocity from Simulation of Apollo 8 Splashdown Drag Data

$v_{spashdn}(t) := 4400 \frac{m}{s} - \int_{0}^{t} a_{Sim}(t) \, dt \cdot s$ $\quad v_{spashdn}(119.9) = 1.607 \cdot \frac{km}{s}$ $\quad 14.8 \cdot 1000 \frac{ft}{s} = 4.511 \cdot \frac{km}{s}$

#2. Simulation of Apollo CM Drag Acceleration asim and Resultant Velocity



Simulation with Fixed Altitude is not very accurate.

XXV. Trajectory Sim of Command Module (CM) from Moon to Earth Splashdown

XV Simulation of Apollo Command Module (CM) Trajectory from Moon to Earth Splashdown

This Simulation Uses the Differential Equation Solving Methodology detailed in:
arXiv:1504.07964 "***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

$kg := 1 \quad m := 1 \quad s := 1 \quad N := 1 \quad s := 1 \quad min := 60s \quad hr := 3600s \quad km := 1000m \quad kmps := km \quad kph := \frac{km}{hr}$

$deg := \frac{2\pi}{360} = 0.017 \quad FRAME := 999 \quad t_{end} := \frac{185min}{999} \cdot FRAME + 1$

Length of Simulation

$\mathbf{t} := t_{end} = 3.0836 \cdot hr \quad mph := 0.447 \cdot 10^{-3} kmps$

$v_0 := 3.8km \quad G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg} \quad R_e := 6370km \quad CM_0 := 13600kg \quad kgf := 9.80665N$

Atmospheric Drag, D: Time and Force

$t_1 := 179min \quad \Delta t := 5min \quad t_2 := t_1 + \Delta t \quad t_{drag} := t_1, t_1 + \frac{t_2 - t_1}{700} .. t_2 \quad D_r := CM_0 \cdot 2.5 \cdot kgf \quad D(t) := if(t_1 < t < t_2, D_r, 0kgf)$

Define Gravitational and Dynamics Equations for Earth and CM

$$\begin{pmatrix} m_e & x_{0e} & y_{0e} & x'_{0e} & y'_{0e} \\ m & x_0 & y_0 & x'_0 & y'_0 \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24}\ kg & 0\ m & 0\ m & 0\ kph & 0\ kph \\ 13600kg & -50000\ km & R_e + 12439km & v_0 & 0kph \end{pmatrix}$$

$t_2 = 184 \cdot min \quad t_{end} = 1.11 \times 10^4 \quad CM(t) := CM_0 \quad 95 \cdot 60 = 5.7 \times 10^3$

Given

Set of Differential Guidance Equations for CM

$\mathbf{x}(0s) = \mathbf{x_0} \quad \mathbf{v_x}(0s) = \mathbf{x'_0}$

$$CM(t) \cdot \left(\frac{d}{dt}\mathbf{v_x}(t)\right) = -G \cdot \frac{CM(t) \cdot \mathbf{m_e} \cdot \mathbf{x}(t)}{\left(\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}\right)^3} - \frac{D(t) \cdot \mathbf{y}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}} \qquad \mathbf{v_x}(t) = \mathbf{x'}(t)$$

$\mathbf{y}(0s) = \mathbf{y_0} \quad \mathbf{v_y}(0s) = \mathbf{y'_0}$

$$CM(t) \cdot \left(\frac{d}{dt}\mathbf{v_y}(t)\right) = -G \cdot \frac{CM(t) \cdot \mathbf{m_e} \cdot \mathbf{y}(t)}{\left(\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}\right)^3} + \frac{D(t) \cdot \mathbf{x}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}} \qquad \mathbf{v_y}(t) = \mathbf{y'}(t)$$

Differential Equation Solver

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{v_x} \\ \mathbf{v_y} \end{pmatrix} := Odesolve\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{v_x} \\ \mathbf{v_y} \end{pmatrix}, t, t_{end}\right]$$

$t := 0s, \frac{t_{end}}{7000} .. t_{end} \quad t_{spdn} := 0s, \frac{t_2}{7000} .. t_2 \quad \alpha := 0, \frac{2\pi}{1000} .. 2\pi$

$t_{orb} := t_2, t_2 + \frac{t_2 - t_1}{70} .. t_{end} \quad \mathbf{D} := D(t_{end}) \quad R_{LM}(t) := \sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}$

$D_x(t) := \frac{D(t) \cdot \mathbf{y}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}} \quad D_y(t) := \frac{D(t) \cdot \mathbf{x}(t)}{\sqrt{\mathbf{x}(t)^2 + \mathbf{y}(t)^2}} \quad m_s := CM(t_{end})$

$Y_{dot} := \left(\mathbf{y}(t_2 + 40s) \quad \mathbf{y}(t_{end})\right)^T$

$F_x := -D_x(t_1 + 10s) \quad F_y := -D_y(t_1 + 10s) \quad F_y = -2.901 \times 10^5 \cdot kg \quad \mathbf{D} = 0 \cdot kgf \quad F_x = -16758.5 \cdot kgf \quad X_{dot} := \left(\mathbf{x}(t_2 + 40s) \quad \mathbf{x}(t_{end})\right)^T$

Simulation of Apollo Command Module Trajectory from Moon to Earth Splashdown

Apollo 11 entered the atmosphere at a
height of 121.9 km & speed (s) 11.1 km/s. Sim is a good match.

Results of Simulation for Re-Entry Height and Speed
See Red Band in Below Plot for Atmospheric Entry
where Atmospheric Drag Forces and Parachute Deploys

$h(t) := R_{LM}(t) - R_e$

$R_{LM}(t_1) = 6494 \cdot km$ $\quad h(t_1) = 124.3 \cdot km$

$R_{LM}(t_2) = 6283 \cdot km$ $\quad s(t_1) = 11.011 \cdot kmps$ $\quad s(t_2) = 4.113 \cdot kmps$

$\frac{t_1}{min} = 179$

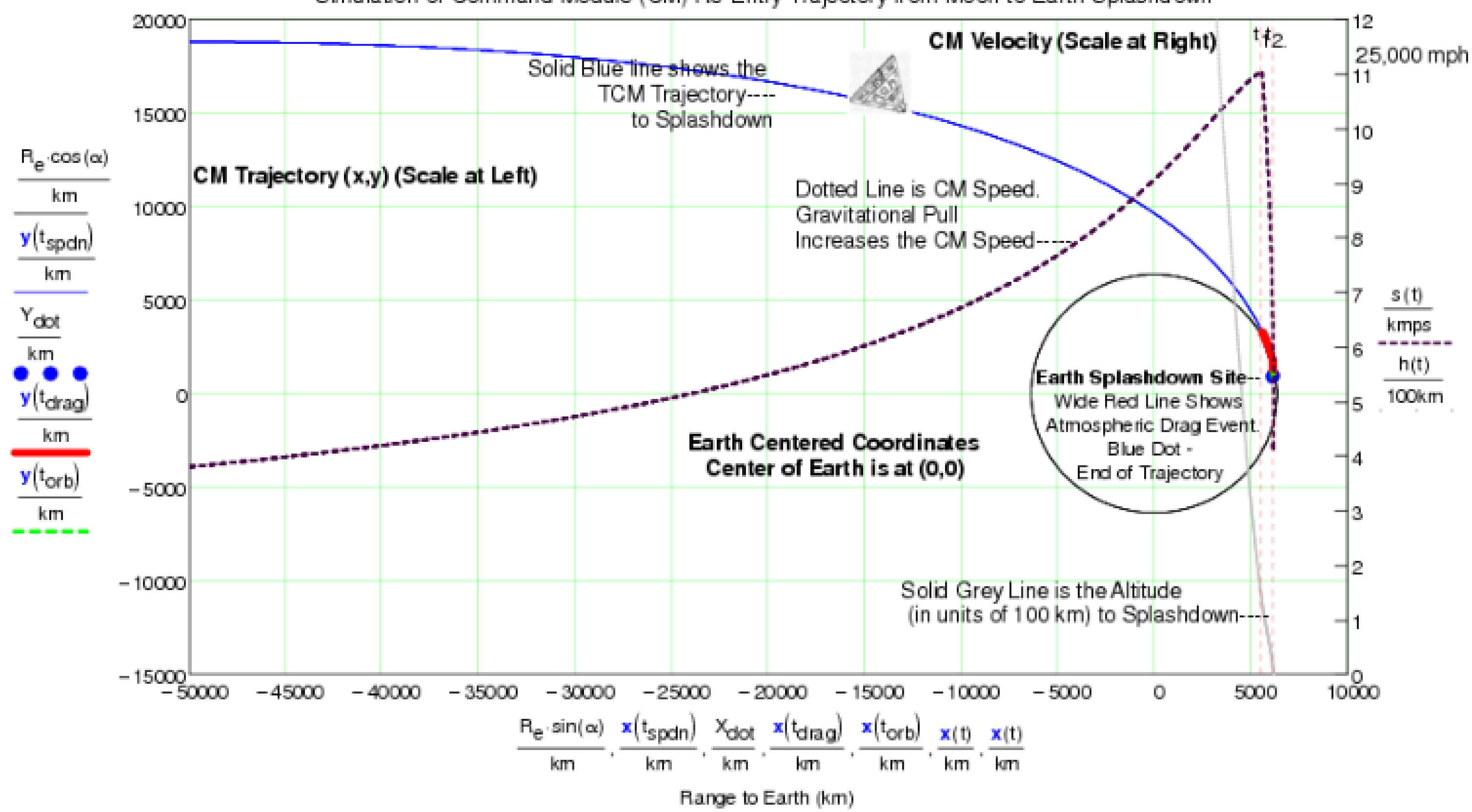
$s(t) := \sqrt{v_x(t)^2 + v_y(t)^2}$ $\quad v_{escape} = 11.182$ kmps

$s(t_2) = 4.113 \cdot kmps$ $\quad s(t_{end}) = 4.281 \cdot kmps$

$s(t_2) = 9.2 \times 10^3 \cdot mph$ $\quad s(t_{end}) = 9.578 \times 10^3 \cdot mph$

$x(t_1) = 5.596 \times 10^3 \cdot km$ $\quad x(t_2) = 6.179 \times 10^3 \cdot km$

Note: The Earth's Atmosphere is only about 100 km thick. The unit squares below are 5000 km.



XXVI. CM Splashdown: Parachute Terminal Velocity

Parachutes were a vital part of the Apollo Mission. They are deployed prior to splashdown for the Apollo command module, CM. The CM could not survive (remain intact) from a hard (high speed) crash into the water, unless slowed down to about 25 mph. For safety and redundancy, the CM had three main parachutes. Terminal Velocity occurs at the velocity where the Drag Force of the parachute equals the force of gravity, or the weight of the CM. Three main parachutes were huge. The amount of material in them was about 1/2 an acre.
The main parachutes were opened by the pilot parachutes at 10,000 feet, and slowed the rate of descent of the command module from 175 miles per hour to 22 miles per hour. In the case of the failure of one parachute (as happened to Apollo 15), the remaining two would be able to decelerate the module to 25 miles per hour. These parachutes held the command module at 27.5 degrees so the module's slanted corner would penetrate the water first, lessening the impact force. After splashdown, the risers of the main parachutes were cut and the parachutes released.

$$A_{chute} := \frac{1}{2}acre$$

$$D_{chute}(v, alt) := \frac{1}{2}\cdot C_d\cdot \rho(alt)\cdot v^2\cdot A_{chute}$$

Terminal velocity at 10,000 ft occurs when:

$$D_{chute} = Weight$$

$$v_{terminal} := \sqrt{\frac{2m_{CM}\cdot g}{0.5\cdot \rho(10ft)\cdot \frac{1}{2}acre}} \qquad v_{terminal} = 39.694\cdot mph$$

At launch the Apollo spacecraft was 363 feet tall and weighed 6.2 million pounds. At splashdown it (the remaining command module) weighs 11,000 pounds and is 36 feet high. Just 0.2% of its original weight and 1/10th the height.

AstroDynamics Glossary and Keplerian Model

In the Keplerian model or elements, named after Johannes Kepler (1571-1630). satellites orbit in an ellipse of constant shape and orientation. The model relates **quantities measured from the earth** to **properties of an ellipse**. The ***key*** to this is that area is swept out at a constant rate in Kepler's model. That is, **(t-T)/A = Period/(π a b),** where T is the time of periapsis passage and A is the area swept out. The Earth is at one focus of the ellipse, not the center (unless the orbit ellipse is actually a perfect circle). The real world is slightly more complex than the Keplerian model, these are compensated by introducing minor corrections or perturbations. the six independent constants defining an orbit that were These constants are:

argument of perigee, ω -- angle from ascending nodes to perigee point along orbit, measured in direction of satellite's motion
eccentricity, e -- defines shape of orbit
inclination angle, i -- gives angle between a satellite's orbital plane and the equator
right ascension of the ascending node -- gives the rotation of orbit plane from reference axis
semi-major axis, a -- defines the max size of orbit
true anomaly, **ν** -- or **θ** defines satellite location on orbit.

Equation of Ellipse. Position r for angle **θ relative to the focus:**

$$r(\theta) = \frac{a(1-e^2)}{1 \pm e\cos\theta}$$

Kepler's Equation: relates an orbit's Mean Anomaly (M) with its Eccentric Anomaly (E):
M(t) = E(t) - e sin (E(t)). Kepler's Equation is transcendental and therefore cannot be solved analytically for E(t). Instead, computers can be used to find the best value of E(t) that satisfies the equation for the known values of M(t) and e. Once E(t) is determined, the True Anomaly, n(t), can be determined. For example: **To compute the position of a point moving in a Keplerian orbit.**
Given: Observation gives that the body passes at (x,y) coordinates x = a(1 − e), y = 0, at time t = t0, then to find out the position of the body at any time, you first calculate the mean anomaly M from the time and the mean motion n by the formula $M = n(t - t_0)$, then compute the value E. This then gives the coordinates: x = a (cos(E) -e) and y = b sin(E)

Angular Momentum of Ellipse, h: Related to the semi-latus rectum, p. **p = h2/μ , where μ = G Me**

Apogee, $r_a = a(1 + e)$
Perigee (Periapsis), $r_p = a(1 - e)$ **,** Point **along the longest axis, a,** (but nearest to the body foci)

Ascending Node (): The precise point in a satellite's orbit that intersects the equatorial plane of the Earth as the satellite moves from the southern to the northern hemisphere (ascending).

Descending Node (☋): The precise point in a satellite's orbit that intersects the equatorial plane of the Earth as the satellite moves from the northern to the southern hemisphere (descending).

Argument of perigee: angle from ascending nodes to perigee point along orbit, measured in direction of satellite's motion. **M(t) = E(t) - e sin (E(t)).**

Astronomical Unit (AU): about 149,599,000 kilometers; the distance from the Earth to the Sun.

azimuth: angular distance in degrees measured in a clockwise direction from true north.

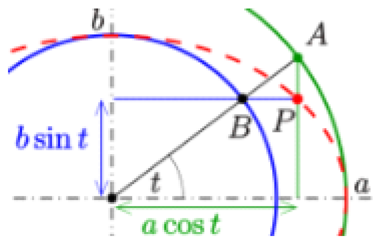
Ephemeris: an arrangement of a series of data points defining both the position and motion of a satellite

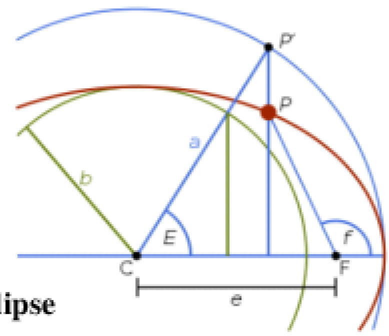
Eccentricity (e): defines how oval the satellite's orbit is. It is mathematically defined as the ratio of the orbit's focus distance (c) to the orbit's semi-major axis **(a). e = c/a**

Eccentric Anomaly (E): The angle, measured since perigee, based on the hypothetical position P' on the **circular orbit** defined by a line perpendicular to the major axis that **passes through the true position P** of the satellite and **intersects with the circular orbit at P'**. The Mean Anomaly is directly related to the Eccentric Anomaly through Kepler's Equation. **M(t) = E(t) e sin (E(t)).**
Ellipse is given by the equation: (x/a)2 + (y/b)2 = 1. Then from the right triangle in Figure: **cos E = x/a. Also, for** (x, y) = (a cos t, b sin t) $0 \leq t < 2\pi$, the parameter t is called the **eccentric anomaly**.

Eccentricity, e: $e = \frac{c}{a}$

c is the distance from the center to either foci, F.
where a and b are defined as semi-major and semi-minor axes





Focus Distance (c): The distance from the **true center** of the orbit **ellipse** to the center of the **Earth**.

Mean Anomaly (M): M was defined by Kepler. It is the angle measured since perigee that would be swept out by the satellite **if** its orbit were **perfectly circular**. It was worked out by geometrical construction. If Pfa equals the Area (Pfa) swept out by the Ellipse from point a to P, then M = Pfa / (1/2 a*b). **This hypothetically constructed orbit** would assume the real orbit's semi-major axis and its period. The Mean Anomaly indicates where the satellite was in its orbit at a specific time.
The mean anomaly is convenient since it is a geometric quantity which is **directly proportional** to the **time**. **M = n * (t - τ),** where τ is the time at the perigee and n = 2π/T. **M(t) = E(t) e sin (E(t))** The **Mean Anomaly, M, equals the True Anomaly, ν,** ***for a circle***. By definition, $\mathbf{M - M_0 = n^* (t - t_0)}$, where M_0 equals the True Anomaly of an ellipse, **ν,** at time 0, and n is the mean motion. It is the fraction of an orbit period that has elapsed since the perigee

Mean Motion (n): Number of orbits the satellite completes about the Earth in exactly 24 hours or where μ is equal to G * Mass. $n = \sqrt{\frac{\mu}{a^3}}$

Period (T): The time required for the satellite to orbit the Earth (or a planet) once.

Perturbations: small adjustments made to the Keplerian model of a satellite's orbit; due to Earth's gravity and drag, a satellite's orbit is not a perfect ellipse of constant shape and orientation.

Semi-Major Axis (a): The distance from the **center** of the orbit ellipse to satellite's apogee or perigee point. This is also defined as the average distance of the satellite from the Earth's center. It can be found from Kepler's 3rd Law: a^3 is proportional to T^3 for n = the Mean Motion of the satellite's orbit.
or by Newton's Laws: $\mathbf{a^3 = GM / (2\pi n)^2}$ or $\mathbf{a^3 = G\,M\,T^2 / 4\pi^2} = kT^2$ where $k = GM / 4\,\pi^2$
The Semi-Major axis, **a** depends only on the Specific Total Energy (energy/mass), $\mathcal{E} = v_0^2 - \mu/r_0^2$ at a
Semi-Minor Axis (b): The shortest distance from the **true center** of the orbit ellipse to the orbit path.

Semi-Latus Rectum, p: $\mathbf{p = a^*(1 - e^2)}$ It is the y coordinate of the ellipse

Time of Flight, t T: Given the above, we can derive for the time of flight $t - T = (E - e \cdot \sin(E)) \cdot \sqrt{\frac{a^3}{G \cdot M}}$

True Anomaly (ν): The true angle, measured since the perigee that the satellite sweeps out while **orbiting the Earth**. See polar equation of an ellipse.